Cryptography

Lecture 6

Announcements

- HW2 due Wednesday, 2/15

Agenda

- Last time:
  - Indistinguishability in the presence of an eavesdropper (K/L 3.2)
  - Defining PRG (K/L 3.3)
- This time:
  - Constructing computationally secure SKE from PRG (K/L 3.3)
  - Security Proof (K/L 3.3)
  - Class Exercise on PRG's

Pseudorandom Generator (PRG)

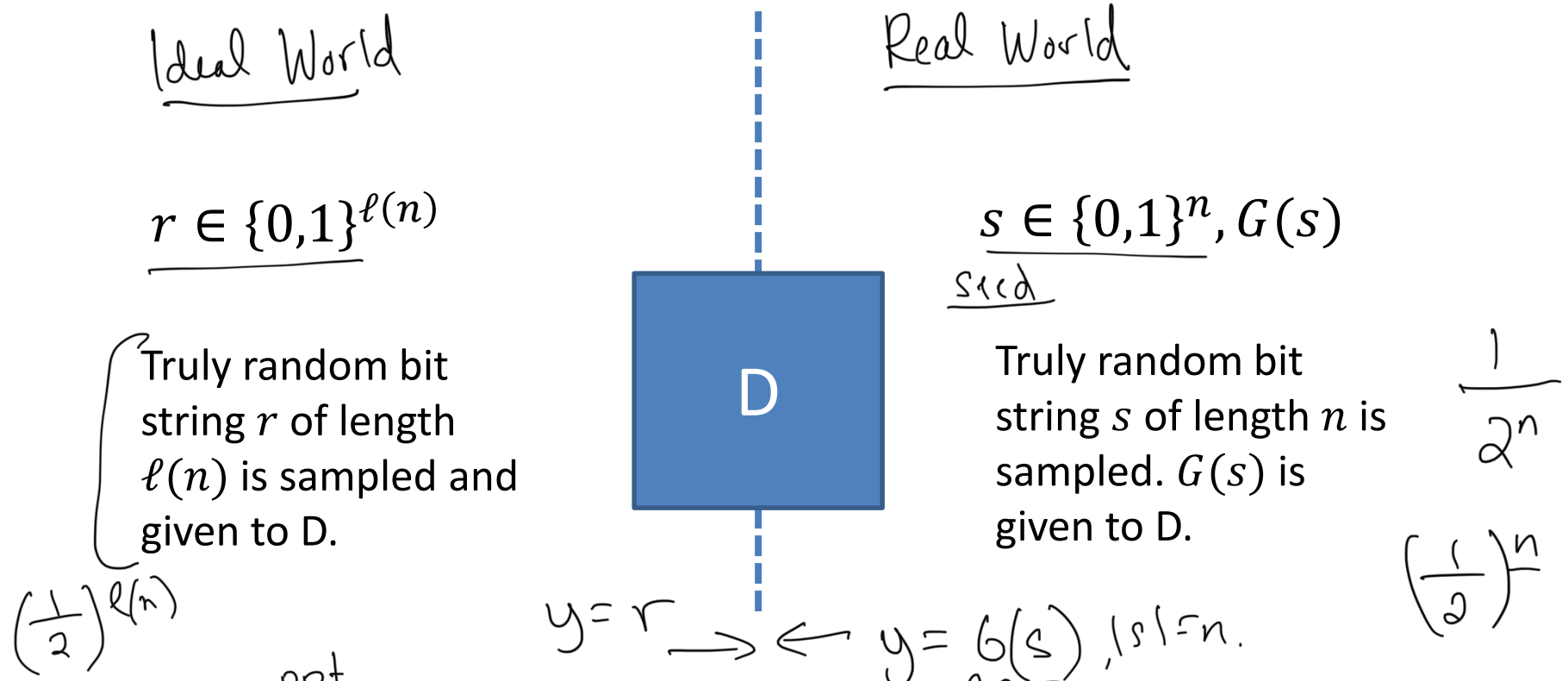
- Functionality
  - Deterministic algorithm G
  - Takes as input a short random seed s
  - Ouputs a long string $G(s)$
- Security
  - No efficient algorithm can "distinguish" $G(s)$ from a truly random string r.
  - i.e. passes all "statistical tests."
- Intuition:
  - Stretches a small amount of true randomness to a larger amount of pseudorandomness.
- Why is this useful?
  - We will see that pseudorandom generators will allow us to beat the Shannon bound of $|\boldsymbol{K}| \geq |\boldsymbol{M}|$.
  - I.e. we will build a computationally secure encryption scheme with $|\boldsymbol{K}| < |\boldsymbol{M}|$

bit length of s.

$G(s)$ has input length $|s|$ & output length strictly larger than $|s|$.

$|s|+1$

Pseudorandom Generator (PRG)



Pseudorandom Generators

Definition: Let $\ell(\cdot)$ be a polynomial and let G be a deterministic poly-time algorithm such that for any input $s \in \{0,1\}^n$, algorithm G outputs a string of length $\ell(n)$. We say that G is a pseudorandom generator if the following two conditions hold:

1. (Expansion:) For every n it holds that $\ell(n) > n$.
2. (Pseudorandomness:) For all ppt distinguishers D, there exists a negligible function $negl$ such that:

$$\left|\Pr[D(r) = 1] - \Pr\left[D\left(G(s)\right) = 1\right]\right| \leq negl(n),$$

where r is chosen uniformly at random from $\{0,1\}^{\ell(n)}$, the seed s is chosen uniformly at random from $\{0,1\}^n$, and the probabilities are taken over the random coins used by D and the choice of r and s.

The function $\ell(\cdot)$ is called the expansion factor of G.

Constructing Secure Encryption Schemes

A Secure Fixed-Length Encryption Scheme

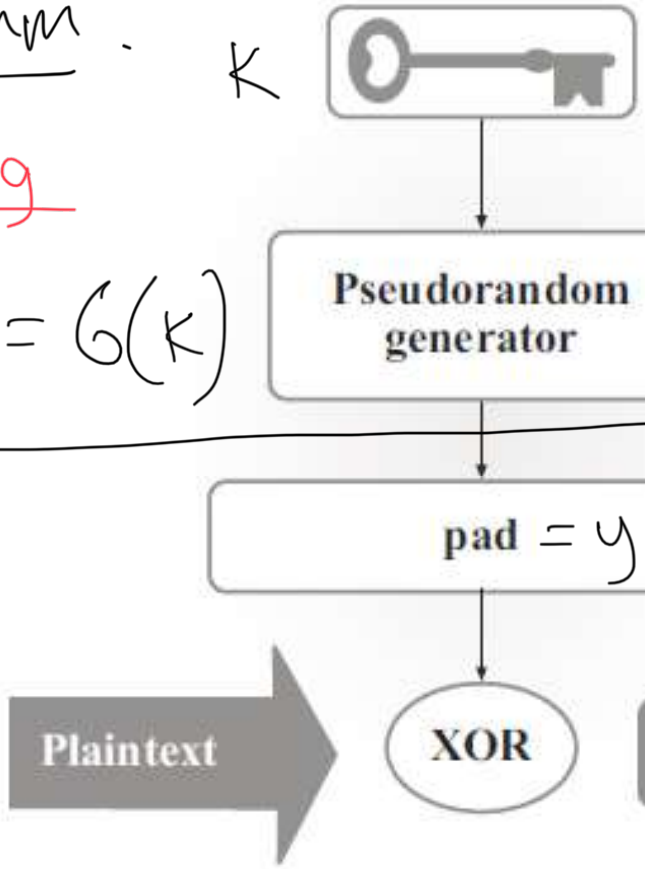
Encrypt Algorithm.

Dec Alg

Construction of a Comp-Secure Enc Scheme from a PRG.

$Gen(1^n) \rightarrow k$

uniformly random key k of length n.



$y = G(k)$

D

$m \in \{0,1\}^{\ell(n)}$

$\ell(n) > n$.

m

c

$c = m \oplus y$.

$m = c \oplus y$

Key space size: 2^n

message space size: $2^{\ell(n)}$

$|M| > |K|$

The Encryption Scheme

Let G be a pseudorandom generator with expansion factor ℓ. Define a private-key encryption scheme for messages of length ℓ as follows:

- Gen: on input 1^n, choose $k \leftarrow \{0,1\}^n$ uniformly at random and output it as the key.
- Enc: on input a key $k \in \{0,1\}^n$ and a message $m \in \{0,1\}^{\ell(n)}$, output the ciphertext
$$c := G(k) \oplus m.$$
- Dec: on input a key $k \in \{0,1\}^n$ and a ciphertext $c \in \{0,1\}^{\ell(n)}$, output the plaintext message
$$m := G(k) \oplus c.$$

Security Analysis

Theorem: If G is a pseudorandom generator, then the Construction above is a fixed-length private-key encryption scheme that has indistinguishable encryptions in the presence of an eavesdropper.

High level: Proofs by reductions $\neg q \rightarrow \neg p$.

If the construction is NOT secure then G is NOT a PRG.

Indistinguishability in the presence of an eavesdropper

Definition: A private key encryption scheme $\Pi = (Gen, Enc, Dec)$ has indistinguishable encryptions in the presence of an eavesdropper if for all probabilistic polynomial-time adversaries A there exists a negligible function $negl$ such that

$$\Pr\left[PrivK^{eav}_{A,\Pi}(n) = 1\right] \leq \frac{1}{2} + negl(n),$$

Where the prob. is taken over the random coins used by A, as well as the random coins used in the experiment.

Pseudorandom Generators

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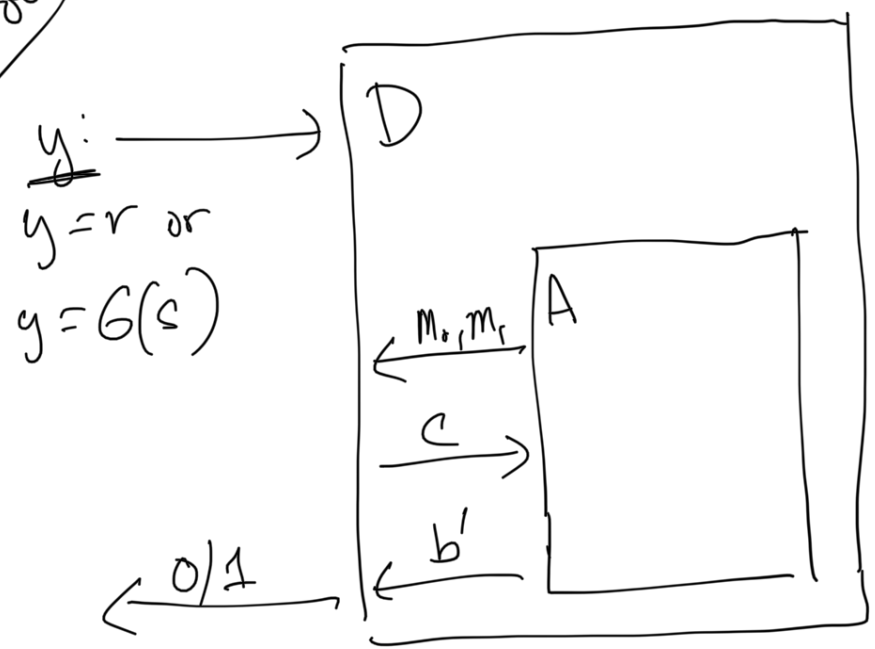
1. (Expansion:) For every n it holds that $\ell(n) > n$.
2. (Pseudorandomness:) For all ppt distinguishers D, there exists a negligible function $negl$ such that:

$$\left|\Pr[D(r) = 1] - \Pr\left[D\left(G(s)\right) = 1\right]\right| \leq negl(n),$$

where r is chosen uniformly at random from $\{0,1\}^{\ell(n)}$, the seed s is chosen uniformly at random from $\{0,1\}^n$, and the probabilities are taken over the random coins used by D and the choice of r and s.

The function $\ell(\cdot)$ is called the expansion factor of G.

Proof - Build a distinguisher D out of an adversary A



Specify: 1. How does D generate c ("challenge")

a. choose $b \xleftarrow{R} \{0,1\}$

b. output $c = m_b \oplus y$

2. Upon receiving b' from A, what does D output

a. Check if $b' = b$

b. If yes, output 1 o/w output 0.

$$\Pr[D(G(s)) = 1] = \Pr[PrivK^{eav}_{A,\Pi}(n) = 1]$$

by assumption $\geq \frac{1}{2} + \rho(n)$ (ρ is non-negl

$$\Pr[D(r) = 1] = \frac{1}{2} \quad \text{(b/c it is a one-time pad)}.$$

$$\left| \Pr[D(r) = 1] - \Pr[D(G(s)) = 1] \right| \geq \epsilon(n)$$

$\epsilon(n)$ is non-negligible by assumption.

$\lightning$

Security Analysis

- Proof by reduction method.

Security Analysis

Proof: Let A be a ppt adversary trying to break the security of the construction. We construct a distinguisher D that uses A as a subroutine to break the security of the PRG.

Distinguisher D:

D is given as input a string $w \in \{0,1\}^{\ell(n)}$.

1. Run $A(1^n)$ to obtain messages $m_0, m_1 \in \{0,1\}^{\ell(n)}$.
2. Choose a uniform bit $b \in \{0,1\}$. Set $c := w \oplus m_b$.
3. Give c to A and obtain output b'. Output 1 if $b' = b$, and output 0 otherwise.

Security Analysis

Consider the probability D outputs 1 in the case that w is random string r vs. w is a pseudorandom string $G(s)$.

- When w is random, D outputs 1 with probability exactly ½. Why?
- When w is pseudorandom, D outputs 1 with probability $\Pr\left[PrivK^{eav}{}_{A,\Pi}(n) = 1\right] = \frac{1}{2} + \rho(n)$, where ρ is non-negligible.

Security Analysis

D's distinguishing probability is:

$$\left|\frac{1}{2} - \left(\frac{1}{2} + \rho(n)\right)\right| = \rho(n).$$

This is a contradiction to the security of the PRG, since ρ is non-negligible.