# TECHNICAL RESEARCH REPORT

Consensus Problems on Small World Graphs: A Structural Study

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**CSHCN TR 2006-3**
**(ISR TR 2006-10)**

Chapter 1

# Consensus Problems on Small World Graphs: A Structural Study

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Consensus problems arise in many instances of collaborative control of multi-agent complex systems; where it is important for the agents to act in coordination with the other agents. To reach coordination, agents need to share information. In large groups of agents the information sharing should be local in some sense, due to energy limitation, reliability, and other constraints. A consensus protocol is an iterative method that provides the group with a common coordination variable. However, local information exchange limits the speed of convergence of such protocols. Therefore, in order to achieve high convergence speed, we should be able to design appropriate network topologies. A reasonable conjecture is that the small world graphs should result in good convergence speed for consensus problems because their low average pairwise path length should speed the diffusion of information in the system. In this paper we address this conjecture by simulations and also by studying the spectral properties of a class of matrices corresponding to consensus problems on small world graphs.

[1] The material is based upon work supported by National Aeronautics and Space Administration under award No NCC8235.

## 1.1 Introduction

Consensus problems arise in many instances of collaborative control of multi-agent complex systems; where it is important for the agents to act in coordination with the other agents. [10, 4, 8, 5]. In this paper we consider Vicsek's model for leaderless coordination and reaching consensus [4, 10], in which at each time instant each agent's state variable is updated using a local rule based on the average of its own state variable plus the state variables of its neighbors at that time. The local neighborhoods are time dependent in general. Each agent's dynamic can be represented as:

$$\theta_i(t+1) = <\theta_i(t)> = \frac{1}{1+n_i(t)}[\theta_i(t) + \sum_{j\in N_i(t)} \theta_j(t)] \tag{1.1}$$

Here $N_i(t)$ denotes the set of neighbors of agent $i$ at time $t$ and $n_i(t)$ denotes the cardinality of this set. The dynamics of the system can be written in matrix form. Let $\mathbb{G}_u$ be the set of possible graphs on $n$ vertices. Let $P$ be a suitably defined set that indexes the set $\mathbb{G}_u$ and $p \in P$. For each $G_p \in \mathbb{G}_u$ define a corresponding $F$-matrix as:

$$F_p = (I + D_p)^{-1}(A_p + I) \tag{1.2}$$

where $A_p$ is the adjacency matrix of the graph $G_p$ and $D_p$ is the diagonal matrix whose $i^{\text{th}}$ diagonal element is the degree of vertex $i$.

This way the simplified Vicsek's model is represented as a switched linear system whose switching signal takes values in a set of indices that parameterize the set of underlying graphs.

$$\theta(t+1) = F_{\sigma(t)}\theta(t) \tag{1.3}$$

The $F$ matrices are a class of stochastic matrices and convergence of consensus protocols depends on properties of their infinite products. In this way linear consensus schemes are closely related to Markov chains and random walks on graphs with self loops. Different connectivity assumptions (symmetric vs. asymmetric neighborhoods) as well as different topology assumptions (fixed vs. changing) result in different sufficient conditions for convergence of consensus problems which can be found in [4, 3, 1] and references therein. In this paper we limit our scope to symmetric topologies, for which being connected is a sufficient condition for convergence. In fact there exist even less restrictive assumptions for convergence.

This paper addresses the convergence speed and effects of structural properties of graphs on performance of consensus protocols. After discussing measures of convergence speed, we study the convergence of consensus protocols for a class of complex networks, known as "small world" graphs [12], leading us to propose design guidelines for reaching consensus fast. We examine the conjecture that dynamical systems coupled in this way would display enhanced signal propagation and global coordination, compared to regular lattices of the same size. The intuition is that the short paths between distant parts of the network cause high speed spreading of information which may result in fast global coordination.

The organization of the paper is as follows. First, we use Perron-Frobenius theory of nonnegative matrices to show that the Second Largest Eigenvalue Modulus (SLEM) of the corresponding $F$ matrices are a good measure of convergence of the consensus protocol. Then we study the convergence speed for small world graphs and try to find design guidelines. We use simulations to show a drastic improvement of convergence speed by considering small values of $\phi$ and use graph spectral methods to reason about this behavior.

## 1.2 Speed of convergence in fixed and changing topologies

A very important issue in consensus problems is the speed of convergence. The faster the consensus is reached, the better the performance of the protocol. Since the applications that use consensus protocols involve many agents, it is necessary for all of them to converge quickly. The convergence rate is a function of the topology of the underlying graphs. This problem is actually in close connection with the asymptotic behavior of Markov chains. In fact if we consider a fixed topology, the convergence rate of the consensus protocol is nothing but the convergence rate to the stationary distribution of the Markov chain corresponding to the stochastic matrix $F$. Consider the system:

$$\theta(t+1) = F_{\sigma(t)}\theta(t) \tag{1.4}$$

as before where $F_p = (I+D_p)^{-1}A_p$ are stochastic matrices with nonzero diagonal elements. In the case of fixed graph topology, the second largest eigenvalue modulus (SLEM) of the corresponding $F$ matrix determines the convergence speed. This is because,

$$\theta(\infty) - \theta(t) = (F^{\infty} - F^t)\theta(0) \tag{1.5}$$

Since $F$ is a primitive stochastic matrix, according to the Perron-Frobenius theorem [9], $\lambda_1 = 1$ is a simple eigenvalue with a right eigenvector $\mathbf{1}$ and a left eigenvector $\pi$ such that $\mathbf{1}^T\pi = 1$, $F^{\infty} = \mathbf{1}\pi^T$ and if $\lambda_2, \lambda_3, ..., \lambda_r$ are the other eigenvalues of $F$ ordered in a way such that $\lambda_1 = 1 > |\lambda_2| \geq |\lambda_3| \geq ... \geq |\lambda_r|$, and $m_2$ is the algebraic multiplicity of $\lambda_2$, then

$$F^t = F^{\infty} + O(t^{m_2-1}|\lambda_2|^t) = \mathbf{1}\pi^T + O(t^{m_2-1}|\lambda_2|^t) \tag{1.6}$$

where $O(f(t))$ represents a function of $t$ such that there exists $\alpha, \beta \in R$, with $0 < \alpha \leq \beta < \infty$, such that $\alpha f(t) \leq O(f(t)) \leq \beta f(t)$ for all $t$ sufficiently large. This shows that the convergence of the consensus protocol is geometric, with relative speed equal to SLEM. We denote $\mu = 1 - SLEM(G)$ as the spectral gap of a graph, so graphs with higher spectral gaps converge more quickly.

For the general case where topology changes are also included, Blondel *et al* [1] showed that the joint spectral radius of the set of matrices determines the convergence speed. For $\Sigma$ a set of finite $n \times n$ matrices, their joint spectral radius is defined as:

$$\rho = \limsup_{t \to \infty} \max_{A_1,\ldots,A_t \in \Sigma} ||A_t \ldots A_1||^{1/t} \tag{1.7}$$

Calculation of the joint spectral radius of a set of matrices is a mathematically hard problem and is not tractable for large sets of matrices. Our goal is to find network topologies which result in good convergence rates. Switching over such topologies will also result in good convergence speed. We limit our scope to the case of fixed topology and examine the conjecture that "small world" graphs have high convergence speed.

## 1.3 Convergence in "small world" graphs

Watts and Strogatz [11] introduced and studied a simple tunable model that can explain behavior of many real world complex networks. Their "small world" model takes a regular lattice and replace the original edges by random ones with some probability $0 \leq \phi \leq 1$. It is conjectured that dynamical systems coupled in this way would display enhanced signal propagation and global coordination, compared to regular lattices of the same size. The intuition is that the short paths between distant parts of the network cause high speed spreading of information which may result in fast global coordination. We examine this conjecture. To the best of our knowledge, the only existing result in the literature is a very recent paper of Olfati-Saber [7], which belongs to continuous time consensus protocols and contains some conjectures on the second largest eigenvalue of the Laplacian of the small world graphs. In this study, we use a variant of the Newman-Moore-Watts [6] improved form of the $\phi-$model originally proposed by Watts and Strogatz. The model starts with a ring of $n$ nodes, each connected by undirected nodes to its nearest neighbors to a range $k$. Shortcut links are added -rather than rewired- between randomly selected pairs of nodes, with probability $\phi$ per link on the underlying lattice; thus there are typically $nk\phi$ shortcuts. Here we actually force the number of shortcuts to be equal to $nk\phi$ (comparable to the Watts $\phi-$model.) In our study, we have considered different initial rings $(n, k) = (100, 2), (200, 3), (500, 3), (1000, 5)$, generated 20 samples of small world graphs $G(\phi)$ for 50 different $\phi$ values chosen in a logarithmic scale between 0.01 and 1. Picking these choices of $(n, k)$ is done for comparison purposes with the results of [7]. In the figures 1.1 and 1.2, we have depicted the gain in spectral gap of the resulting small world graphs with respect to the spectral gap of the base lattice. We will just include the results of cases $(500, 3)$ and $(1000, 3)$. The others follow a similar pattern. Some important observations and comments follow:

1. In the low range of $\phi$ $(0 < \phi < 0.01)$ there is no spectral gap gain observed and the SLEM is almost constant and a drastic increase in the spectral gap is observed around $\phi = 0.1$.

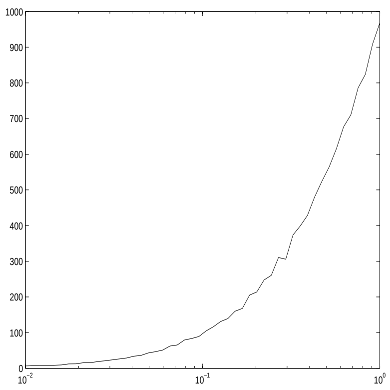


**Figure 1.1**: Spectral gap gain for $(n, k) = (500, 3)$

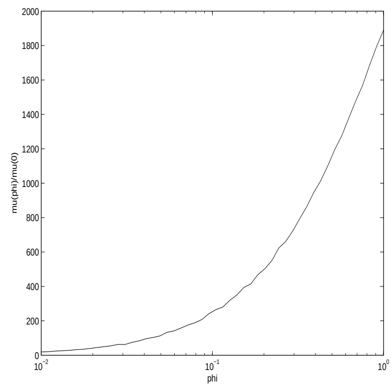


**Figure 1.2**: Spectral gap gain for $(n, k) = (1000, 5)$

2. Simulations show that "small world graphs" possess good convergence properties as far as consensus protocols are concerned. Some analytical results are included in the next section but the complete analysis as will be mentioned is subject to future work.

The results show that adding $nk\phi$ shortcuts to a $1 - d$ lattice dramatically improves the convergence properties of consensus schemes for $\phi \approx 0.1$. For example in a $(500, 3)$ lattice, by adding randomly 150 edges, we can on average increase the spectral gap approximately by a factor of 100. However, our aim is to find a more clever way of adding edges so that after adding 150 edges to a $(500, 3)$ lattice we get much more increase in the spectral gap.

To formulate this problem, we consider a dynamic graph which evolves in time starting from a 1-d lattice $G_0 = C(n, k)$. Let's denote the complete graph on $n$ vertices by $K_n$. Also, denote the complement of a graph $G = (V, E)$ -which is the graph with the same vertex set but whose edge set consists of the edges not present in $G$ - by $\bar{G}$. So, $E(\bar{G}) = E(K_n) \setminus E(G)$.

If we denote the operation of adding an edge to a graph by $A$, the dynamic graph evolution can be written as:

$$\begin{array}{ll} G(t+1) = A(G(t), u(t)) & t = 0, 1, 2, ..., nk\phi - 1 \\ u(t) = e(t+1) & e(t+1) \in E(\bar{G}(t)) \\ G(0) = G_0 & \end{array} \tag{1.8}$$

So, now the problem to solve is:

$$\begin{array}{c} \max_{e(1),\dots,e(n) \in E(\bar{G}(t))} \max\left[\lambda_2(F(nk\phi)), -\lambda_N(F(nk\phi))\right] \\ subject \quad to: \quad (1.8) \end{array} \tag{1.9}$$

where $F(nk\phi) = D(G(nk\phi))^{-1}A(G(nk\phi))$. We will now mention some observations which are useful to build a framework for studying the above problem.

### 1.3.1 Spectral analysis

The choice of $G_0 = C(n,k)$ to be a regular 1-d lattice with self loops means that (possibly after re-labeling vertices) the adjacency matrix of the graph can be written as a circulant matrix:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & . & . & . & a_n \\ a_n & a_1 & a_2 & . & . & . & a_{n-1} \\ a_{n-1} & a_n & a_1 & . & . & . & a_{n-2} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_2 & a_3 & . & . & . & . & a_1 \end{pmatrix} = circ[a_1, a_2, ...a_n] \tag{1.10}$$

in which:

$$a \triangleq [a_1, a_2, ...a_n] = [\underbrace{1, ..., 1}_{k+1} \quad \underbrace{0, ..., 0,}_{n-2k-1} \quad \underbrace{1, ..., 1}_{k}] \tag{1.11}$$

Circulant matrices have a special structure which provides them with special properties. All entries in a given diagonal are the same. Each row is determined by its previous row by a shift to right (modulo $n$). Consider the $n \times n$ permutation matrix, $\Pi = circ[0 \quad 1 \quad 0 \quad ... \quad 0]$. Then for any circulant matrix we can write: $A = circ[a_1, a_2, ..., a_n] = a_1 I + a_2 \Pi + ... + a_n \Pi^{n-1}$. For a vector $a = [a_1, a_2, ..., a_n]$, the polynomial $p_a(z) = a_1 + a_2 z + a_3 z^2 + ...a_n z^n$ is called the representer of the circulant. The following theorem based on [2] states how to calculate the eigenvalues of circulants.

**Theorem 1.3.1** *[2] Let $\omega = e^{\frac{2\pi\sqrt{-1}}{n}}$ be the nth root of unity. The eigenvalues of $A = circ[a_1, a_2, ..., a_n]$ are given by $\lambda_i = p_a(\omega^{i-1})$, where $i = 1, 2, ...n$.*

The main result considering the spectral properties of $G_0$ follows.

**Proposition 1.3.1** *The corresponding $F$ matrix of $G_0 = C(n,k)$ is circulant. Furthermore, its SLEM has multiplicity at least 2.*

**Sketch of Proof:** Since $G_0 = C(n,k)$ is $2k+1$-regular (including the self loop), $F = D^{-1}A = \frac{1}{2k+1}A$. So $F$ is circulant $F = circ(\frac{1}{2k+1}a)$, where $a$ is as in (1.11). The representer of this circulant is

$$p_a(z) = \frac{1}{2k+1}(1 + z + ... + z^{k-1} + z^k + z^{n-k} + z^{n-k+1} + ... + z^{n-1}) \tag{1.12}$$

So, the eigenvlues of this matrix are $\lambda_i = p_a(\omega^{i-1})$. It is easy to show that $\lambda_1 = 1$ and moreover it is a simple eigenvalue because the underlying graph is connected. Since for integers $A$ and $B$, $\omega^{An+B} = \omega^B$, it follows that $\lambda_2 = \lambda_n$, $\lambda_3 = \lambda_{n-1}$ and so on. In the case that $n$ is odd apart from $\lambda_1 = 1$, all eigenvalues come in pairs. In the case that $n$ is even, it can be shown that $\lambda_{\frac{n}{2}+1}$ is the only eigenvalue apart from $-1$ which can be single, however direct calculation shows that it is equal

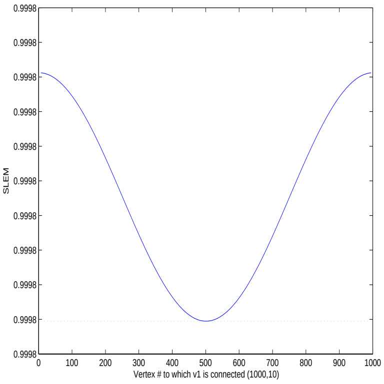


**Figure 1.3**: Adding a shortcut (1000,5), The dotted line tangent to curve shows SLEM before adding edge

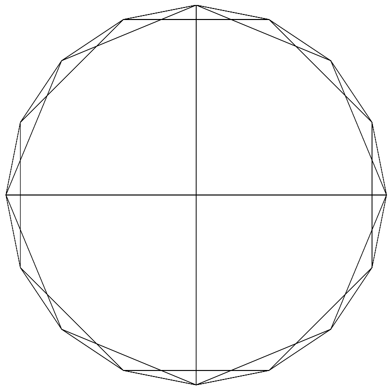

**Figure 1.4**: The optimal topology; adding 2 shortcuts to $C(16, 2)$

to $\frac{(-1)^k}{2k+1}$ which is clearly less than $\lambda_2 = \lambda_n$. A simple geometric argument shows that $SLEM = \lambda_2 = \lambda_n = \frac{1}{2k+1}[1 + 2Re(\omega) + 2Re(2\omega) + ... + 2Re(k\omega)] < 1$ and $\lambda_i \leq \lambda_2$ for $i \in 2, ..., n-1$. This shows that for the case where $k \ll n$, which are the cases we are more interested in, as $n \to \infty$ two of the non-unity eigenvalues approach 1. This describes the slow convergence of consensus protocols when diameter is large. The following theorem gives an upper bound on the change in SLEM resulting from adding shortcuts.

**Theorem 1.3.2** *If $G_l$ is the graph resulted from adding $l$ shortcuts to $G_0 = C(n, k)$ in a way that each vertex involves in at most 1 shortcut, then* $|SLEM(G_l) - SLEM(G_0)| \leq \frac{l}{(k+1)(2k+1)}$

**Sketch of Proof:** Let $F_0$ and $F_l$ show the $F$ matrices corresponding to $G_0$ and $G_l$. $\sum_{i=1}^{n} \lambda_i(G_0) = Tr(F_0) = \frac{n}{2k+1}$ and $\sum_{i=1}^{n} \lambda_i(G_l) = Tr(F_l) = \frac{2l}{2k+2} + \frac{n-2l}{2k+1}$, and the bound follows by subtracting the corresponding eigenvalues due to the above equations.

## 1.4 Simulation results: The effect of adding one and two shortcuts

We ran a set of simulations with different purposes based on (1.8). A counter intuitive result is that the SLEM does not monotonically change with addition of edges. Specially, in cases when $n$ is even, adding an edge will increase SLEM unless the case that a vertex is connected to the farthest vertex from it that is $i$ is connected to $i + n/2$ (modulo 2). In this case one of the multiplicities of the SLEM is lessened but the other multiplicity is not changed. Figures 1.3 and 1.4

illustrate this effect. The dotted line tangent to the curves show the SLEM of the original curves. The more distant the two joined vertices, the less increase in SLEM. Adding two shortcuts can however decrease the SLEM. It is worthwhile to mention that in all of our simulations, for a given $n$, shortcuts that reduced the diameter of the graph more, resulted in higher spectral gap. For example, for the case of adding 2 shortcuts to $G_0 = C(16, 2)$, Figure 1.4 shows the optimal topology. The analysis of this conjecture is subject of future work.

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