# NETWORK TOMOGRAPHY: NEW RIGOROUS APPROACHES FOR DISCRETE AND CONTINUOUS PROBLEMS

*John S. Baras*

The Institute for Systems Research, University of Maryland, College Park, MD 20742, USA.

**ABSTRACT**

We consider several rigorously defined network tomography problems from applications ranging from communication networks, to social networks. The universal abstraction we develop involves the inference of various network structural and parametric properties form observations of certain "probing" processes from a subset of network nodes which we call the boundary nodes of the network. We show that these problems lead to mathematical problems of "deconvolution" over unconventional semirings, inversion of integrals over trees of the underlying graph, Radon transform over symmetric spaces, and completion of sparse matrices albeit in non-conventional semirings. We illustrate applications in several applied problems. Some of the problem formulations and solutions are inspired by generalizations of "electric impedance problems" in various non-nonconventional spaces. Some of the solutions have a variational interpretation.

***Index Terms—*** network tomography, adaptive probes, electrical impedance tomography

## 1. INTRODUCTION

Networks and networked systems are ubiquitous in all aspects of life and work. Today's communication (and other networks) are increasingly larger and heterogeneous, the prime example being the Internet, but also corporate, or government networks. Sensor networks are becoming ubiquitous and they are envisioned to consist of large numbers of interconnected heterogeneous sensors gathering and transmitting information about the weather, road traffic, structural integrity of bridges, buildings, human and animal patterns, etc.. For proper and efficient network management network monitoring and behavior understanding are critical requirements. Monitoring is also essential for protecting the network from malicious attacks, for preventing catastrophic failures, for allocating resources, and for delivering specified quality of service (QoS). Network monitoring is even more demanding in mobile wireless networks. It is clearly impractical to monitor each node or link of the network separately. Sophisticated methods of active network probing or passive traffic monitoring can generate network statistics that indirectly relate to the performance measures we require.

Thus many practical problems associated with networks lead to the following prototypical "inverse" problem: Estimate the values of some critical parameters or variables associated with the links and nodes of the network, or detect the occurrence of come critical events, using only information that is collected from a small subset of the nodes of the network. Additional problems are inference of network structure and behavior. Typically the measurements are collected via measurements and probing processes from this set of nodes, which we call the *boundary nodes* of the network. There are many examples including the identification of bottlenecks in communication networks, identification of faulty lines and equipment in communication networks, medical diagnostics, monitoring of power grids, monitoring of social and economic networks over the Web, identification of matches between customer (user) profiles and product characteristics, distributed collaborative inference and control in mobile sensor and robotic networks, and many others. The variables that need to be estimated, as well as the observable variables, can be logical or numerical, leading to interesting hybrid problems. We will refer to these problems collectively as *Network Tomography* problems. The most challenging problems of network tomography are those that require minimal cooperation from network elements that cannot be directly controlled. The vast majority of the methods considered and investigated todate in network tomography are statistical [1, 2, 3]. The relation to true mathematical tomography is absent. In addition the methods used todate are weak in that they do not take advantage of structural or logical information available on the network or the traffic.

Most of the approaches to address these problems utilize some heuristics and they perform extensive evaluations of these heuristic methods and algorithms. A large variety of these problems remarkably share some important common characteristics regardless of the specific application. They can be transformed [4] to the prototypical *electrical impedance tomography (EIT)* problem in its continuous [5, 6, 7] or discrete form [8, 9, 10, 11, 12]. This is the approach we describe in this paper. It has the advantage that it leads to a systematic modeling approach to many of these problems, and at the same time it helps create very innovative and high performance algorithmic solutions. In a very significant departure from current methods our approach formulates these network tomography problems rigorously as mathematical tomography problems on networks and their representation as graphs or their embeddings in various homogeneous spaces [13].

## 2. DISCRETE ELECTRICAL IMPEDANCE TOMOGRAPHY

This is classical problem and it leads to several powerful generalizations. Following [12] we consider resistor networks in the plane. For each positive integer $n$, a square network $\Omega$ is constructed as follows. The nodes of $\Omega$ are the integer lattice points in the plane $p = (i; j)$ for which $0 \leq i \leq n + 1$ and $0 \leq j \leq n + 1$; the four corners are excluded. The set of

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nodes is denoted $\Omega_0$. The interior of $\Omega_0$, called $int\Omega_0$, consists of those nodes $p = (i; j)$ with $1 \leq i \leq n$ and $1 \leq j \leq n$. The boundary of $\Omega_0$, called $\partial\Omega_0$, is $\Omega_0 - int\Omega_0$. Each interior node $p$ has four neighboring nodes which are the nodes at unit distance from $p$; the set of four neighbor nodes is called the *neighborhood* of $p$, denoted by $\mathcal{N}(p)$. Each interior node has all of its neighbors in $\Omega_0$. Each boundary node $p$ has exactly one neighboring node which is the interior node at unit distance from $p$. An edge $pq$ of $\Omega_0$ is the horizontal or vertical line segment which connects a pair of neighboring nodes $p$ and $q$ in $int\Omega_0$, or which connects a boundary node $p$ to its neighboring interior node $q$. The set of edges is denoted $\Omega_1$. An edge $pq$, where $p$ is a boundary node and $q$ is its neighboring interior node is called a *boundary edge*. A *network of resistors* is a network $\Omega = (\Omega_0, \Omega_1)$ together with a positive real-valued function $\gamma$ on $\Omega_1$. For each edge $pq$ in $\Omega_1$, the number $\gamma(pq)$ is called the *conductance* of $pq$, and $1/\gamma(pq)$ is the *resistance* of $pq$. The function $\gamma$ is called the *conductivity*. A function $u : \Omega_0 \to R$ gives a current across each conductor $pq$ by *Ohm's law:* $I = \gamma(pq)(u(p) - u(q))$. A function $u : \Omega_0 \to R$ is called $\gamma - harmonic$ if for each interior node $p$,

$$\sum_{q \in \mathcal{N}(p)} \gamma(pq)(u(q) - u(p)) = 0. \qquad (1)$$

This property of a $\gamma$-harmonic function is really *Kirkhoff's law*. If a function $\varphi$ is defined at the boundary nodes, the network $\Omega$ will acquire a unique $\gamma$-harmonic function $u$, with $u(p) = \varphi(p)$ for each boundary node $p$; the *potential* due to $\varphi$. Now, in turn the function $u$ determines a current $I_\varphi(p)$ through each boundary node $p$, by $I_\varphi(p) = \gamma(pq)(u(p) - u(q))$, where $q$ is the interior neighbor of $p$. For each conductivity $\gamma$ on $\Omega_1$, the linear map $\Lambda_\gamma$ from boundary functions to boundary functions is defined by $\Lambda_\gamma\varphi = I_\varphi$. The boundary function $\varphi$ is called Dirichlet data, the boundary current $I_\varphi$ is called Neumann data, and the map $\Lambda_\gamma$ is called the Dirichlet-to-Neumann map. The *inverse problem* of interest is to recover the conductivity $\gamma$ from $\Lambda_\gamma$; and leads immediately to four natural problems:

(1) *Uniqueness*: If $\Lambda_\gamma = \Lambda_\mu$, does it follow that $\gamma = \mu$?

(2) *Continuity*: If $\Lambda_\gamma$ is near $\Lambda_\mu$, is $\gamma$ near $\mu$?

(3) *Reconstruction*: Algorithm to compute $\gamma$ using $\Lambda_\gamma$.

(4) *Characterization*: For each integer $n$, which $4n$ by $4n$ matrices are of the form $\Lambda_\gamma$ for some $\gamma$?

This set of problems, and various generalizations constitute the simplest form of network tomography; this particular example is discrete electric impedance tomography (EIT) [12]. The four problems have been answered in the affirmative, by duality both the Dirichlet-to-Neumann and the Neumann-to-Dirichlet map uniquely determine the conductivity, efficient algorithms for computing the solution have been developed and the space of solutions to the fourth problem has been efficiently described and parametrized [8, 9, 10, 11, 12]. Additional problems have been investigated, hat are of greater interest to the theme of this paper, such as fast determination of faulty resistors, shorts and other faults in the resistor network. The innovative idea that these methods and algorithms employ is that of *adaptive probing*. That is a small set of boundary notes is used at each step, for partial reconstruction of the conductivity (on a subset of interior edges), and then based on this information and reconstruction a new set of boundary nodes is used to extend the reconstruction to more edges [8, 9, 10, 11, 12]. Interest has continued strong and there has been a continuous output of applications and exciting new algorithms and applications till today. This discrete problem is the discrete version of the famous Calderón's problem [5, 6, 7, 4], which is related to numerous inverse problems ranging form radar scattering, to medical imaging to clocking [4], and which has attracted enormous attention and work through the years.

## 3. FROM ELECTRICAL TO GENERAL NETWORKS: RADON TRANSFORMS ON TREES AND GRAPHS

In communication networks, tomography aims to reconstruct interior link parameters (which play the role of electrical network conductances) based on end-to-end measurements. These measurements are based on observing packets that go through the network from source to destination boundary nodes. Depending on what type of link parameters we wish to reconstruct, we could measure end-to-end packet delays, packet interarrival times, packet losses, or packet flow bandwidth. In the MINC project [2, 1] multicast probe packets are used from a source to multiple destinations. A set of $n$ packets is sent to $R$ multicast receivers, and each receiver records whether each packet was received or lost. After all packets are sent, each receiver $j$ has recorded a size $n$ vector $r_j$ of binary variables. Assuming loss probabilities $\alpha_e$ for each internal edge $e$, we can compute the likelihood of a vector $\alpha$ of loss probabilities (one for each edge), given the observed fate of each packet. We call $\Theta = \{0, 1\}^R$ the set of values that the vectors $x_i$ can get, and we denote by $n(x), x \in \Theta$, the number of occurrences of vector $x$. The $x_i$ vectors are used to form the log-likelihood function $\mathcal{L}(\alpha)$ and the aim is to find the vector $\alpha$ that maximizes it [2, 1]. This is done by introducing some intermediate variables $\gamma_k$ that are defined as follows: $k$ is a node of the multicast tree, and $\gamma_k$ is the probability that at least one of the receivers "under" $k$ (i.e. the path from the source to the receiver goes through $k$) receives a packet. They estimate $\gamma_k$ by the fraction of packets that are received by at least one receiver under $k$. They then proceed to calculate the vector $\alpha$ from the set $\gamma_k$.

This is similar to the electrical impedance problem. In a manner of speaking, we just have packets instead of electrons. The analogy is clearer if we think back to the random walks analogy. Instead of fractions of packets that reach a receiver we have the probability that a boundary node will be reached by the random walk. We send a unit (up to the maximum possible) flow of traffic and we observe how much of it reaches each receiver. This will depend on the internal link capacities and is related to the link saturation problem that we intend to solve. By increasing the flow from the source to the receivers, we will not see an increase of incoming flow at those receivers that are reachable only through paths with at least one saturated link. If we use the $r_j$ vectors, we can do a similar calculation, which is equivalent to inverting the discrete Radon transform on trees [14, 15, 16]. We can see this by taking the logarithm of the likelihood equations for each receiver. This problem is similar to the delay inference problem, which is exactly what is solved by the inverse Radon transform [17, 18, 13]. These questions have not been answered for more general networks. Does uniqueness continue to hold for general graphs? What is the Dirichlet-to-Neumann map in that case? What are the probes, the equivalent of the potential and the current function?

The analogy between tree graphs and Radon transform in hyperbolic space has been established and fruitfully exploited [15, 16, 14, 19, 17]. It has been solved explicitly in the case of trees. Can the general case be solved by embedding the graph in a hyperbolic space of higher dimension? Trees are planar graphs, so they can be embedded in a unit disk, which is the 2-dimensional plane in hyperbolic geometry. General graphs should be embeddable in a higher dimensional hyperbolic plane (ball of dimension $n$).

### 3.1. The weighted graph model

In this case we model our network in the following way. We have a collection of nodes and edges between the nodes in a finite planar connected graph $G$. We denote by $V$ the set of nodes of $G$ and by $E$ the set of edges of $G$. Usually, the graph G is denoted by $G(V, E)$. A particular subset of this graph $G$ is denoted by $\partial G$ and called the boundary of $G$. In our context these are the nodes accessible to whoever is trying to monitor the traffic in $G$. The boundary edges are those links whose two endpoints are in $\partial G$. We assume that $G$ remains connected even if we remove the boundary edges. For our present purposes, the boundary edges play no role, thus we may as well assume there are none. We also assume that $\partial G$ is not empty.

Furthermore, we assume that to every edge in $E$ we have an associated non-negative number $\omega(x, y)$ which corresponds to the traffic between the endpoints $x$ and $y$ of the edge. We define the degree $d_\omega x$ of a node $x$ in the weighted graph $G$ with weight $\omega$ by

$$d_\omega x = \sum_{y \in V} \omega(x, y).$$

The Laplacian operator corresponding to this weight $\omega$ is

$$\Delta_\omega f := \sum_{y \in V} [f(y) - f(x)] \cdot \frac{\omega(x, y)}{d_\omega x}, \ x \in V$$

A graph $S = S(V', E')$ is said to be a *subgraph* of $G(V, E)$ if $V' \subset V$ and $E' \subset E$. In this case, we call $G$ a *host graph* of $S$. The integration of a function $f : G \to \mathbb{R}$ on a graph $G = G(V, E)$ is defined by

$$\int_G f = \sum_{x \in V} f(x) d_\omega x \text{ or simply } \int_G f d_\omega$$

For a subgraph $S$ of a graph $G = G(V, E)$ the (node) *boundary* $\partial S$ of $S$ is defined to be the set of all nodes $z \in V$ not in $S$ but adjacent to some node in $S$, i.e.,

$$\partial S = \{z \in V \mid z \sim y \text{ for some } y \in S\}$$

and we define the *inner boundary* $\overset{\circ}{\partial} S$ by

$$\overset{\circ}{\partial} S := \{z \in S \mid y \sim z \text{ for some } y \in \partial S\}$$

where $z \sim y$ means that the two nodes $z$ and $y$ are connected by an edge in $E$. Also, by $\overline{S}$ we denote a graph whose nodes and edges are in $S \cup \partial S$. The (outward) normal derivative $\frac{\partial f}{\partial_\omega n}(z)$ at $z \in \partial S$ is defined to be

$$\frac{\partial f}{\partial_\omega n}(z) := \sum_{y \in S} [f(z) - f(y)] \cdot \frac{\omega(z, y)}{d'_\omega z},$$

where $d'_\omega z = \sum_{y \in S} \omega(z, y)$

In this model, there are two kinds of disruptions of traffic data that could arise. In one of them, disruptions occur when an edge "ceases" to exist; in this case, the topology of the graph has changed, and we refer to the important work of Fan Chung and her collaborators which offers crucial insights into this question [20]. In the other, the weights change because of "increase" in traffic. That is, the network configuration remains the same but the weights have either increased or remained the same. In this second situation, we can appeal to the following theorem [19].

**Theorem 1** *Let $\omega_1$ and $\omega_2$ be weights with $\omega_1 \leq \omega_2$ on $\overline{S} \times \overline{S}$ and $f_1, \ f_2 : \overline{S} \to \mathbf{R}$ be functions satisfying that for $j = 1, \ 2$,*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0, \ x \in S, \\ \frac{\partial f}{\partial_{\omega_j} n}(z) = \psi(z), \ z \in \partial S, \\ \int_S f_j \ d_{\omega_j} = K \end{cases}$$

*for a given function $\psi : \partial S \to \mathbf{R}$ with $\int_{\partial S} \psi = 0$ and for a suitably chosen number $K > 0$. If we assume that*

*(i)* $\omega_1(z, y) = \omega_2(z, y)$ *on* $\partial S \times \overset{\circ}{\partial} S$,

*(ii)* $f_1|_{\partial S} = f_2|_{\partial S}$ ,

*then we have*

$$f_1 = f_2 \text{ on } \overline{S}$$

*and*

$$\omega_1 = \omega_2 \quad \text{on } \overline{S} \times \overline{S}$$

*whenever $f_1(x) \neq f_1(y)$ and $f_2(x) \neq f_2(y)$*

We conclude that the data distinguishes the two cases. That is, we can decide whether there is an increase of traffic somewhere in the network or not. While this is only a uniqueness theorem, nevertheless, we can effectively compute the actual weights from the knowledge of the Dirichlet data for convenient choices of the input Neumann data in a way similar to that done for lattices. Similarly, the Green function of this Neumann boundary value problem can be represented by an explicit matrix.

What we want to discuss now is the relationship between the above results and the problem of understanding a large network like the Internet. One way to make more concrete this problem was discussed by T. Munzner in [21] on visualizing the Internet. It implies that the natural domain might be a hyperbolic space of dimension higher than 2. One can see that Munzner's suggestion leads to a question closely resembling EIT, and it is natural to consider it a problem in hyperbolic tomography [15, 16, 14, 19, 17]. The inversion of the Neumann-Dirichlet problem is better to be fomulated directly on "weighted" graphs [19]. Similarly, the Radon transform in the hyperbolic plane has been studied extensively [15, 16, 14, 13, 17].

For the sake of completeness, we will describe here a simplified version of the Radon transform on trees and its inversion formula. As explained below, this seems to be enough to deal with the network problems we are interested in.

### 3.2. The Radon transform on homogeneous trees and its inversion

We consider trees $T$. Given two vertices $u$ and $v$, we say they are neighbors if $(u, v)$ is an edge and write $u \sim v$

in this case. A geodesic $\gamma$ from $u_0$ to $u_l$ is a collection $u_0, u_1, \ldots, u_{l-1}, u_l$ of pairwise distinct vertices such that $u_0 \sim u_1, u_1 \sim u_2, \ldots, u_{l-1} \sim u_l$. To simplify the notation, for any geodesic $\gamma = u_0 \sim u_1 \sim u_2 \sim \ldots \sim u_{l-1} \sim u_l$ we denote by $-\gamma$ the geodesic with the opposite orientation, i.e., $-\gamma = u_l \sim u_{l-1} \sim \ldots \sim u_0$. The collection of all (open) geodesics is denoted by $\Gamma$. If $T$ is infinite, then a complex valued function $f \in L^1(T)$ if $\sum_{v \in V} |f(v)| < \infty$. The Radon transform $R$ of a function $f \in L^1(T)$ is simply the bounded function $Rf$ on $\Gamma$ defined by $Rf(\gamma) = \sum_{v \in \gamma} f(v)$. Under mild conditions, the Radon transform in a tree is invertible. In fact, the explicit inversion formula resembles that of the inversion for the Radon transform in the Euclidean plane [14, 17, 13]. It is easier to compute or homogeneous trees.

**Theorem 2** *The identity $R^*R = \beta\mu_0 + \sum_{n=1}^{\infty} 2\beta\mu_n$ holds in $L^1(T)$, providing a bounded extension of $R^*R$ to $L^2(T)$.*

**Theorem 3** *The unique bounded extension to $L^2(T)$ of the operator $R^*R$ is invertible on $L^2(T)$, with inverse the operator*

$$E = \frac{2(q+1)^3}{q(q-1)^2}\left[\mu_0 + \sum_{n=1}^{\infty}(-1)^n 2\mu_n\right]$$

*which is convolution with $\frac{2(q+1)^3}{q(q-1)^2}\left[h_0 + \sum_{n=1}^{\infty}(-1)^n 2h_n\right]$.*

**Theorem 4** *The Radon transform $R : L^1(T) \longrightarrow L^{\infty}(\Gamma)$ is inverted by $ER^*Rf = f$*

To invert $R$, observe that $R^*R$ acts as a convolution operator with a known radial kernel $h = \beta h_0 + \sum_{n=1}^{\infty} 2\beta h_v$.

## 4. CONCLUSIONS

We described several formulations of inference problems in networks, as mathematically rigorous tomography problems. We showed that many of these tomography problems can be transformed to discrete or continuous EIT problems. Therefore the methods available for EIT problems, most notably Radon transforms and inversions become applicable to these problems and lead to efficient algorithms. This may require embedding the networks in appropriate homogeneous spaces including higher dimensional hyperbolic spaces. Another very [promising feature of our approach is that the resulting algorithms quite often have a variational interpretation [22] which provides evidence for, and methods to make them more robust.

## 5. REFERENCES


[1] N. Duffield, "Simple network performance tomography," in *Proc. ACM SIGCOMM Internet meas. conf.*, 2003.

[2] T. Bu, N. Duffield, F. Lo Presti, and D. Towsley, "Network tomography on general topologies," in *Proc. ACM SIGMETRICS*, 2002.

[3] S. Bohacek and B. Rozovsky, "Network tomography," in *Proc. IEEE systems, man, and cybernetics information assurance workshop*, 2001.

[4] G. Uhlmann, "Electrical impedance tomography and calderón's problem," *Inverse Problems*, vol. 25, no. 12, pp. 123011, 2009.

[5] A. P. Calderon, "On an inverse boundary value problem," *Sem. Num. An. and its Appl. to Continuum Physics, Soc. Brazileira de Mathematica*, pp. 65–73, 1980.

[6] R. Kohn and M. Vogelius, "Identification of an unknown conductivity by means of measurements at the boundary," *SIAM-AMS Proceedings*, vol. 14, pp. 113–123, 1985.

[7] J. Sylvester and G. Uhlmann, "A global uniqueness theorem for an inverse boundary value problem," *Ann. of Math.*, vol. 125, pp. 153–169, 1987.

[8] E. B. Curtis, T. Edens, and J. A. Morrow, "Calculating the resistors in a network," in *Proc. Ann. Int. Conf. IEEE Engin. in Med. Biol. Soc.*, 1989, vol. 11, pp. 451–452.

[9] G. F. Lawler and J. Sylvester, "Determining resistances from boundary measurements in finite networks," *SIAM J. Discrete Math.*, vol. 2, pp. 211–239, 1989.

[10] E. B. Curtis and J. A. Morrow, "Determining the resistors in a network," *SIAM J. Appl. Math.*, vol. 50, pp. 918–930, 1990.

[11] E. B. Curtis and J. A. Morrow, "The Dirichlet to Neumann map for a resistor network," *SIAM J. Appl. Math.*, vol. 51, no. 4, pp. 1011–1029, 1991.

[12] E. B. Curtis and J. A. Morrow, *Inverse problems for electrical networks*, World Scientific, 2000.

[13] S. Helgason, *The Radon transform*, Birkhauser, 1999.

[14] C. A. Berenstein and E.C. Tarabusi, "Integral geometry in hyperbolic spaces and electrical impedance tomography," *SIAM J. Appl. Math.*, vol. 56, pp. 755–764, 1996.

[15] J. S. Baras, C. A. Berenstein, and F. Gavilanez, "Continuous and discrete inverse conductivity problems," *AMS Contemporary Math. J.*, vol. 362, pp. 33–51, 2004.

[16] C. A. Berenstein, F. Gavilanez, and J. S. Baras, "Network tomography," *AMS Contemporary Math. J.*, vol. 405, pp. 11–17, 2006.

[17] S. Lissianoi and I. Ponomarev, "On the inversion of the geodesic Radon transform on the hyperbolic plane," *J. inverse problems*, vol. 13, pp. 1053.

[18] F. Santosa and M. Vogelius, "A backprojection algorithm for electrical impedance imaging," *SIAM J. Appl. Math.*, vol. 50, no. 1, pp. 216–243, 1990.

[19] C. A. Berenstein and S.-Y. Chung, "$\omega$-harmonic functions and inverse conductivity problems on networks," *SIAM J. Appl. Math.*, vol. 65, no. 4, pp. 1200–1226, 2005.

[20] F. Chung, "Distance realization problems with applications to internet tomography," *JCSS*, vol. 63, no. 3, pp. 432–448, 2001.

[21] T. Munzner, "Exploring large graphs in 3d hyperbolic space," *IEEE comp. Graphics and Appl.*, 1998.

[22] A. Bensoussan and J. L. Menaldi, "Difference equations on weighted graphs," *J. Convex Analysis*, vol. 12, no. 1, pp. 13–44, 2005.