# THE CONSENSUS PROBLEM UNDER VANISHING COMMUNICATIONS*

CHRISTOFOROS SOMARAKIS† AND JOHN S. BARAS ‡

**Abstract.** We revisit the classical multi-agent distributed consensus problem under the dropping of the assumption that the existence of a connection between agent implies weights uniformly bounded away from zero. We formulate and study the problem by establishing global convergence results in discrete time, under fixed, switching and random topologies. We study the application of the results to flocking networks.

**Key words.** Linear processes on graphs, consensus, coefficient of ergodicity, flocking

**AMS subject classifications.** 05C50 , 05C90 , 93C05 , 60J10 , 34D05

## 1. Introduction and Motivation.

Consensus problems arise in many instances of collaborative control of multi-agent complex systems; where it is important for the agents to act in a distributed, whereas coordinated, manner, [14, 7, 10], to name a few. However, in the vast majority of all relevant works the exchange of information among any two communicating nodes occurs under an established connection with a weight that is uniformly bounded away from zero. This rudimentary assumption ensures the applicability of a large number of analytical tools from linear algebra, algebraic graph theory, probability theory discussed in the literature of the control community.

When running a distributed algorithm on the network the main consensus result (i.e. agreement of all agents' states) suggests that an assumption of strong connectivity in principle ensures convergence to a common value through a repeated convex averaging of states. The uniform bound of weights implies a geometric time rate of convergence [12].

In this paper, we revisit the classical consensus problem with $k$ agents in $\mathbb{R}^n$

$$z_i(t+1) - z_i(t) = \sum_{j=1}^{k} a_{ij}(t)\big(z_j(t) - z_i(t)\big) \quad i = 1, \dots, k$$

where $a_{ij}(t) : \mathbb{Z}_+ \to \mathbb{R}_+$ are $C^0$ functions defined in the following. We study the time asymptotic behaviour of $z_i \in \mathbb{R}^n$.

At first, we discuss the elementary complete connectivity static case from which we built up to more interesting variations of the problem such as switching signals as well as random failures, where we study and establish results for global convergence. In fact, the results suggest that the rate at which weights can vanish should not be faster than a critical value which depends on the topology of the graph. We outline the motivation of this paper with the following example

*Example.* Consider the 2-D system

$$\begin{pmatrix} x(t+1) \\ y(t+1) \end{pmatrix} = \begin{pmatrix} 1-f(t) & f(t) \\ g(t) & 1-g(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

*This work was partially supported by the Army Research Office under MURI grant W911NF-08-1-0238, the US Air Force Office of Scientific Research MURI grant FA9550-09-1-0538, and by the National Science Foundation (NSF) grant CNS-1035655

†The Institute For Systems Research, University Of Maryland, College Park,(csomarak@umd.edu).

‡The Institute For Systems Research, University Of Maryland, College Park, (baras@umd.edu)

where $f(t) = K_f/t^2$ and $g(t) = K_g/t^2$. $t \geq 1$. Then for $|x(0) - y(0)| = \delta \neq 0$

$$|x(t+1) - y(t+1)| = |1 - f(t) - g(t)||x(t) - y(t)| = \delta \prod_{i=0}^{t} (1 - f(i) - g(i))$$
$$\to C \sin(\pi \sqrt{K_f + K_g}) \quad \text{as} \quad t \to \infty$$

for some $C > 0$ according to the Euler-Wallis formula. So for $\sqrt{K_f + K_g} \notin \mathbb{Z}_+$ consensus is not achieved.

The paper is organized as follows. In section 2, we present the necessary notations and definitions as well as the main analytical tools we will use. In section 3, we prove the main results of this work and discuss important generalizations of the discrete time problem. In section 4, we discuss two variants of the same problem. The stochastic verstion of the problem as well as an application in flocking dynamics where the connection with the Cucker-Smale model is discussed. Finally in section 5, we revise the results and make some concluding remarks.

**2. Notations and Definitions.** We consider the $k$ agents in a Euclidean space. As usual $\mathbb{Z}, \mathbb{R}$ are the sets of integers and reals respectively. By $|| \cdot ||_p$ we denote the $p$-vector norm and $|\cdot|_\infty$ the infinity norm in $\mathbb{R}^k$, $k \in \mathbb{N}$. All the vectors are considered column. By $\mathbb{1}$ we understand the $k-$ dimensional vector with all entries equal to 1.

By a *weighted undirected graph* $\mathbb{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ where $\mathcal{V}$ is the set of vertices, $\mathcal{E} = \{(i,j) : i,j \in \mathcal{V}\}$ the set of edges and $\mathcal{W} = \{w_{ij} : (i,j) \in \mathcal{E}\}$ the set of weights on edges. The *degree* $l = l_i$ of a node $i$ is the number of adjacent edges to $i$. The graph $\mathbb{G}$ is connected if for any two vertices $i, j$ there is a path of edges $(l_k, l_{k-1})|_{k=0}^{m}$ such that $l_0 = i$ and $l_m = j$. The matrix representation of a graph $\mathcal{G}$ of $k$ vertices is done with the use of the $k \times k$ adjacency matrix $A = [w_{ij}]$, the diagonal matrix $D = [d_{ii}]$ where $d_{ii}$ is the degree of node $i$, and the Laplacian matrix defined by $L := D - A$.

The *agreement* or *consensus* space $\mathcal{C}$ is defined as the subset of $\mathbb{R}$ such as

$$\mathcal{C} = \{x_i \in \mathbb{R} : x_1 = x_2 = \cdots = x_k\}$$

A *rank* - 1 matrix, $A = [a_{ij}]$, is such that it has identical rows. It follows that $Ax \in \mathcal{C}\ \forall x \in \mathbb{R}^n$.

A *non-negative (positive)* matrix $A = [a_{ij}]$ is such that $a_{ij} \geq 0 (> 0)$. A *stochastic* matrix is a non-negative matrix $A = [a_{ij}]$ such that $\sum_j a_{ij} = 1\ \forall i$, so $\lambda = 1$ is an eigenvalue and $\mathbb{1}$ is the corresponding left eigenvector. The Perron-Frobenius theorem discusses the asymptotic behaviour of powers of non-negative matrices [12], i.e. homogeneous products of matrices. In this paper, we will need notions from the theory of non-homogeneous products of matrices [12, 3]. The main tool for studying such products of matrices is the coefficient of ergodicity.

**2.1. The Coefficient Of Ergodicity.** A measure of measuring the contraction rate of nonnegative matrices is the coefficient of ergodicity.

DEFINITION 1. *For a nonnegative matrix $A = [a_{ij}]$, the quantity*

$$\tau(A) := \max_{||z||_1 = 1, z'\mathbb{1} = 0} ||A'z|| = \frac{1}{2} \max_{i,j} \sum_l |a_{il} - a_{jl}| = 1 - \min_{i,j} \sum_l \min\{a_{il}, a_{jl}\} \quad (2.1)$$

*is called the coefficient of ergodicity of $A$.*

One can think of $\tau$ either as a vector norm maximized over the disagreement space $\mathbb{R}^n \backslash \mathcal{C}$ or as an eigenvalue bound expressed in terms of a deflated matrix, with the deflation approximating the dominant spectral projector. There are many different definitions and names for $\tau$, all applicable for different spaces. The following proposition introduces the basic properties of $\tau$.

PROPOSITION 1 ([12, 3]). *For $A$ and $B$ stochastic matrices we have the following properties:*

1. $0 \leq \tau(A) \leq 1$. $\tau(A) < 1$ *if and only if every pair of rows $\alpha, \beta$ of $A$ have a common position $k$ such that $a_{\alpha k} a_{\beta k} > 0$. $\tau(A) = 0$ if and only if $A$ is a rank - 1 matrix.*
2. $\tau(AB) \leq \tau(A)\tau(B)$ *(Sub-multiplicativity property).*

So $\tau$ is a norm that measures the amount of contraction of an operator matrix over $\mathbb{R}^n \backslash \mathcal{C}$ and recognizes a matrix with identical rows. The family of matrices that are generally "recognizable" by $\tau$, (i.e. $A$ stochastic such that $\tau(A) < 1$) play an important role.

DEFINITION 2. *The stochastic matrix, $A$, such that $\tau(A) < 1$, is called scrambling.* Using the definition of $\tau$ we have the following important remark

REMARK 1. *A stochastic matrix, $A$, is scrambling if given two rows $i, j$ there is at least one column $k$ such that $a_{ik} > 0$ and $a_{jk} > 0$. Equivalently in such a matrix, no two rows are orthogonal.*

PROPOSITION 2 ([12]). *If the graph representation of $A$ is strongly connected, then there exists $\gamma \geq 1$ such that $A^\gamma$ is scrambling.*

The following Theorem is useful when $A$ is scrambling.

THEOREM 1 ([12]). *Let $w$ be a non-negative vector and $A$ a stochastic matrix. If $z = Aw$ then*

$$\max_i z_i - \min_i z_i \leq \tau(A)\big(\max_i w_i - \min_i w_i\big) \tag{2.2}$$

**3. The discrete time problem.** Consider $k$ agents lying in a $n$-dimensional Euclidean space. At each time each agent will update it's status $z_i(t) \in \mathbb{R}^n_+$ according to

$$z_i(t+1) - z_i(t) = \sum_{j=1}^{k} a_{ij}(t)\big(z_j(t) - z_i(t)\big) \qquad i = 1, \ldots, k \tag{3.1}$$

ASSUMPTION 1. *The connectivity functions as functions of time are defined as follows $a_{ij}(t) : \mathbb{Z}_+ \to \mathbb{R}_+$ such that*

$$\begin{aligned} &(i) \qquad a_{ij}(t) \leq a < \frac{1}{k-1} \\ &(ii) \qquad a_{ij}(t) \in \omega(t^{-\alpha}) \end{aligned} \tag{3.2}$$

*where $\alpha \geq 0$, $t$ is time and $k$ the number of agents.*

DEFINITION 3. *We say that a function $f(t)$ is asymptotically dominant to $g(t)$ as $t \to a$ and write $f(t) \in \omega(g(t))$ if $\lim_{t \to a} f(t)/g(t) = \infty$.*

We understand that in our case, the limit implies that $\forall\ i, j \in \mathcal{V}$, $\exists \mathcal{T} : a_{ij} \geq ct^{-\alpha}$, $\forall\ t \geq \mathcal{T}$, and in our case we also demand $c = c(k) < \mathcal{T}^\alpha/(k-1)$, so that Assumption 1(i) is satisfied. This in general implies that weights are free to vanish asymptotically as $t^{-\alpha}$, but not faster.

For any $i$ we rewrite (3.1) as

$$z_i(t+1) = \big(1 - \sum_{j=1}^{k} a_{ij}(t)\big) z_i(t) + \sum_{j=1}^{k} a_{ij}(t) z_j(t)$$

or in the matrix form

$$z(t+1) = (G(t) \otimes I_d)z(t) \tag{3.3}$$

where $G(t) := I_k - L_x$, $I_k$ is the $k$-dimensional identity matrix and $L_x$ is the Laplacian $L_x = D_x - A_x$. Also, $\otimes$ is the Kronecker product. Since for $A, B, C, D$ matrices $(A \otimes B)(C \otimes D) = AC \otimes DB$ we conclude that we can simply work in the one-dimensional Euclidean space, i.e. take $n = 1$. The following lemma is trivial and helps justify the assumption $z_i \in \mathbb{R}_+$.

LEMMA 1. *Given any vector $w$ and a stochastic matrix $A$, $Aw$ can be written as the sum of a nonnegative vector and a nonpositive constant vector.*

*Proof. For any column vector $w$ trivially write $w = (w + \min_i\{w_i\}\mathbb{1}) - \min_i\{w_i\}\mathbb{1} := u + z$. Obviously $u$ is nonnegative and since the effect of any such $z$ is $A$-invariant the result follows.* $\square$

From (3.3) we end up with $z(t+1) = G(t)z(t)$ where

$$G(t) := \begin{pmatrix} 1 - \sum_{j \neq 1} a_{1j} & a_{12} & \cdots & a_{1k} \\ a_{21} & 1 - \sum_{j \neq 2} a_{2j} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & 1 - \sum_{j \neq k} a_{kj} \end{pmatrix} \tag{3.4}$$

Given the initial vector $z(0)$, the general solution at time $t$ is

$$z(t) = G(t-1)G(t-2) \cdots G(0)z(0)$$

DEFINITION 4. *We say that the system* (3.3) *converges to unconditional consensus if for any initial vector $z(0)$ we have that $\inf_{\bar{z} \in \mathcal{C}} ||z(t) - \bar{z}||_p \to 0$ as $t \to \infty$.*

The root to convergence passes through the dynamics of backward products of non-homogeneous matrices with weights that are not uniformly bounded away from zero. The handling of such products of matrices is a rather intricate subject. Seneta in [12] discusses the different notions of ergodicity that appear depending on the products, i.e. whether they are forward or backward. The *weak ergodicity* refers to the case where in a product $T_{p,k} = T_{p+k}T_{p+k-1} \cdots T_{p+1} = [t^{(p,r)}]_{i,j}$ of matrices, we have $t^{p,r}_{i,s} - t^{p,r}_{j,s} \to 0$ as $r \to \infty$ for any $i, j, s, p$. On the other hand, *strong ergodicity* is such that $lim_r t^{(r,p)}_{i,j}$ exists and is independent of $i$. In case of backward products, the case here, the two notions of ergodicity coincide ([12]) and there is no restriction to bounded weights, hence we are consistent with Definition 4.

**3.1. Asymptotic upper bounds for $\tau(G(t))$ .** The definition of $\tau$ requires to find the two rows $i, j$ that minimize the sum $S(i,j) = \sum_{n=1}^{k} \min\{a_{in}, a_{jn}\}$. The structure of $G(t)$ imposes that for every $i, j$ the sum $S(i,j)$ must have three forms depending on the number of diagonal elements that will be included in the sum. In case of one diagonal element $\tau(G(t)) < \delta - \omega(t^{-\alpha})$ for some $0 < \delta < 1$. In case of two diagonal elements inspection yields $\tau(G(t)) < \frac{3}{k-1}$ (the cases $k = 2, k = 3$ are pathological and do not violate generality). Finally, for no diagonal elements $\tau(G(t)) < 1 - \omega(t^{-\alpha})$. We informally proved the following technical lemma:

LEMMA 2. *([6]) Assume $k \geq 2$ agents, which update their speed according to* (3.3)*. Then the coefficient of ergodicity satisfies*

$$\tau(G(t)) \leq 1 - \omega(t^{-\alpha}) \qquad as \qquad t >> 1. \tag{3.5}$$

**3.2. Convergence.**

THEOREM 2. *Under Assumption 1, the system* (3.3) *converges to unconditional consensus* $\forall\alpha \in [0,1]$.

*Proof.* It is enough to show that the maximum coordinate of $z(t)$ approaches the minimum as $t \to \infty$. Set $m(t) := \max_i z_i(t) - \min_j z_j(t)$ and apply Theorem 1 to get

$$m(t+1) \leq \tau(G(t))m(t) \leq \prod_{s=0}^{t} \tau(G(s))m(0),$$

as well as that since $m(t) \geq 0$, $\lim_{t\to\infty} m(t)$ exists. In view of Lemma 1 we calculate:

$$\begin{aligned}\lim_{t\to\infty} m(t) &\leq \lim_{t\to\infty} \prod_{s=0}^{t} \tau(G(s))m(0) \leq C \lim_{t\to\infty} \prod_{s=\mathcal{T}}^{t} (1-\omega(s^{-\alpha}))m(0)\\ &\leq C \lim_{t\to\infty} \prod_{s=\mathcal{T}}^{t} e^{-\omega(s^{-\alpha})} m(0) \leq C \lim_{t\to\infty} e^{-\sum_{s=\mathcal{T}}^{t}\omega(s^{-\alpha})} m(0) = 0\end{aligned}$$

for $0 \leq \alpha \leq 1$ and $\mathcal{T}$ goes according to the definition of the $\omega$ notation and is independent of $t$. ∎

**3.3. Static and Switching Topologies.** The complete connectivity case is rather trivial. In fact, from the definition of $\tau$ we can relax this assumption to cases where $G(t)$ is scrambling. So from now on we drop the Assumption 1(ii) to $a_{ij} \in \omega(t^{-\alpha}) \cup \{0\}$. It can be easily shown that the graph with the minimum number of edges such that $G(t)$ is scrambling is a star graph topology [6]. The following stronger result is derived.

COROLLARY 1. *Under Assumption 1 with* $a_{ij} \in \omega(t^{-\alpha}) \cup \{0\}$ *and some* $B$ *such that for all* $t$ $\{G(i)\}_{i=t+B-1}^{t}$ *contains a scrambling matrix, the system* (3.3) *converges to unconditional consensus* $\forall\alpha \in [0,1]$.

*Proof.* Use Theorem 1 and the fact that by assumption there is a strictly increasing subsequence $\{t_i\}_i$ with $t_1 > 0$, $\lim_i t_i \to \infty$ and $|t_i - t_{i+1}| \leq B$. Then of course $t_i \leq (i-1)B + t_1$ for $i \geq 2$ and

$$\sum_i \frac{1}{t_i^{\alpha}} \geq \sum_i \frac{1}{((i-1)B+t_1)^{\alpha}} = \infty \quad \forall\alpha \in [0,1]$$

∎

Corollary 1, exploits the idea of recurrent "scramblingness", a sort of generalization of recurrent connectivity in the standard literature [14, 7].

The assumption of sole strong connectivity puts the scrambling index into the picture in view of the fact that the product matrix $G(t+\gamma-1)\cdots G(t)$ is scrambling for all $t$.

COROLLARY 2. *Assume a static network with graph* $\mathbb{G} = (V,E)$ *that is strongly connected, so that it has a scrambling index* $\gamma$. *Then* (3.3) *associated with* $\mathbb{G}$ *will reach consensus under the assumptions of Corollary 1, for* $\alpha \in [0, 1/\gamma]$.

*Proof.* At first note that for any $t_1, t_2$, $G(t_1)G(t_2)$ is of the same form as (3.4). From the proof of Lemma 4 it suffices to check the case where the minimum path of elements contains no diagonal element. From this, one may use the observation that $a_{ij}(t_1)a_{lm}(t_2) \geq \omega(t_2^{2\alpha})$ for $t_1 < t_2$, to conclude by induction that,

$$\tau(G(t)G(t+1)\cdots G(t+\gamma-1)) \leq 1 - \omega\big((t+\gamma-1)^{-\gamma\alpha}\big) \qquad t >> 1. \tag{3.6}$$

Then the result follows as in Theorem 1. □

In the case of uniform bounds of weights we have $\alpha = 0$ so that $\gamma$ has no effect. It is an easy exercise to show that $\gamma \leq \frac{k-1}{2}$ so that one can use the recurrent connectivity assumption to obtain new bounds for the critical exponent $\alpha$, which in general would not have to depend on the recurrent connectivity bound, $B$.

**4. Applications.** In this section we will discuss two important variations of the consensus algorithms discussed in the literature [4, 11, 8, 13, 1]

**4.1. Random Graphs.** Consider a probability space $(\Omega_0, \mathcal{B}, \mathbb{P}_0)$, where $\Omega_0$ is the set of $k \times k$ $(0-1)$ matrices with positive diagonals, $\mathcal{B}$ is the Borel $\sigma-$ algebra of $\Omega_0$ and $\mathbb{P}_0$ a probability measure on $\Omega_0$. Define the product probability space $(\Omega, \mathcal{F}, \mathbb{P}) = \prod_k (\Omega_0, \mathcal{B}, \mathbb{P}_0)$ and by the Kolmogorov extension theorem there exists a measure $\mathbb{P}$ that makes its coordinates stochastically independent while preserving the marginal distributions. The elements of the product space have the following forms: $\Omega = \{(\omega_1, \dots, \omega_l, \dots) : \omega_l \in \Omega_0\}$, $\mathcal{F} = \mathcal{B} \times \mathcal{B} \times \cdots$, $\mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_0 \times \cdots$.

It follows that the mapping $W_l : \Omega \to \Omega_0$ is the $l^{th}$ coordinate function, which for all $\omega \in \Omega$ is defined as $W_l(\omega) = \omega_l$ and guarantees that $W_l(\omega)$ $l \geq 1$ are independent random $0-1$ matrices with positive diagonals and common distribution $\mathbb{P}$. Define now $G(W_t, t) : \Omega \times \mathbb{Z}_+ \to G_1$ to be the stochastic matrix $G(t)$ with $0-1$ connectivity between $i$ and $j$ and connectivity weight $a_{ij}$. In the following $G(W_t, t) := G(t)$; we take it as a random variable on the product space such that $G_t$, $G_s$ are independent for $s \neq t$. Then $m(t) = \max_i z_i(t) - \min_j z_j(t)$ is a random variable and we are interested in the probability of the event

$$A_t := \{\limsup_s m(s) = 0 \quad \text{given} \quad m(t) = G(t-1)\cdots G(0)m(0)\}. \tag{4.1}$$

Obviously, $A_1 \supset A_2 \supset \cdots$ and such events are events of the tail $\sigma$-field, $\mathcal{F}_\infty$. By Kolmogorov's $0-1$ law we have that under the independence assumption if $A \in \mathcal{F}_\infty$ then $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$ [2]. This means that we can derive sufficient conditions for almost sure consensus if we derive conditions for the convergence of $m(t)$. The following result is a trivial generalization in the probabilistic frame. Consequently, we are interested in the behaviour of the first moment $\mu(t) = \mathbb{E}[m(t)]$.

THEOREM 3. *The stochastic version of* (3.3) *exhibits unconditional consensus if* $\mathbb{E}[G(t)]$ *is scrambling for all* $t$.

*Proof.* We understand that

$z(t+1) = G(t)z(t) \Rightarrow \mathbb{E}[z(t+1)] = \mathbb{E}[G(t)z(t)] = \mathbb{E}[G(t)]\mathbb{E}[z(t)]$ by independence. So $\mu(t+1) \leq \tau(\mathbb{E}[G(t)])\mu(t)$ and the result follows as in the static case. Given any filtration of $m(k)|_0^t$ we see that

$$\mathbb{E}[m(t+1)|m(0), \dots, m(t)] \leq \tau(\mathbb{E}[G(t)])m(t) \leq (1 - \omega(t^{-\alpha}))m(t)$$

since $\sum_k \omega(k^{-\alpha}) = \infty$ for $0 \leq \alpha \leq 1$ we have that $m(t) \to 0$ with probability 1, as a direct application of the super-martingale convergence theorem [9]. □

**4.2. Flocking Networks.** Another interesting application is in flocking networks. Consider a population of $k$ agents leaving a one-dimensional Euclidean space $\mathbb{E}^d$. At each time $t \in \mathbb{Z}_+$ every agent, $i$, has a vector of state $x_i(t)$ and a vector of velocity $v_i(t)$.

DEFINITION 5. *We say that we have unconditional flocking if* $\forall\ i, j \in \{1, \dots, k\}$ *and all initial positions and velocities, both of the following two conditions hold*

$$(i) \lim_{t\to\infty} ||v_i(t) - v_j(t)|| = 0, \qquad (ii) \sup_{0\leq t<\infty} ||x_i(t) - x_j(t)|| < \infty \qquad \textit{uniformly in } t$$

Assume that every agent adjusts it's velocity by adding to it a weighted average of the differences according to (3.1). Assuming a natural change of position we obtain the system

$$\begin{aligned} x_i(t+1) &= x_i(t) + v_i(t) \\ v_i(t+1) - v_i(t) &= \sum_{j=1}^{k} a_{ij}(t)\big(v_j(t) - v_i(t)\big) \end{aligned} \tag{4.2}$$

Flocking is about speed consensus Def.5 (i), with sufficiently fast rate so that the flock is not destroyed Def.5 (ii). Combination of these two requirements yields that for every pair of $i, j$ agents

$$||x_i(t) - x_j(t)|| \leq ||x_i(0) - x_j(0)|| + \sum_{s=0}^{t-1} ||v_i(s) - v_j(s)||.$$

So it suffices for the speed differences between any two agents to be uniformly summable. The sub-linear growth of $x_i(t)$ implies the sub-linear growth of $||x_i(t) - x_j(t)||$ with time for all $i, j$. Applying Theorem 1 we obtain convergence of $m(t)$ as defined in the proof of Theorem 2, for $\alpha \in [0, 1]$. It only remains to fulfil Def. 5 (ii). Note that for any $t > 0$

$$\sum_{s=0}^{t} m(s) < \sum_{s=0}^{\infty} m(s) \leq \sum_{s=0}^{\infty} \tau(G(s))m(0) \leq B(\mathcal{T}, m(0)) + m(0) \sum_{s=\mathcal{T}}^{\infty} \exp\{-\sum_{j=\mathcal{T}}^{s} \omega(j^{-\alpha})\}.$$

The double sum converges as follows:

$$\begin{aligned} \sum_{s=\mathcal{T}}^{\infty} \exp\{-\sum_{j=\mathcal{T}}^{s} \omega(j^{-\alpha})\} &\leq \sum_{s=\mathcal{T}}^{\infty} \exp\{-\omega(s^{1-\alpha}) + (\mathcal{T}-1)\omega(s^{-\alpha})\} \\ &\leq \exp\{(\mathcal{T}-1)\omega(\mathcal{T}^{-\alpha})\} \sum_{s=\mathcal{T}}^{\infty} \exp\{-\omega(s^{1-\alpha})\} \leq D(\mathcal{T}) \sum_{s=\mathcal{T}}^{\infty} (e^{-c})^{s^{1-\alpha}}. \end{aligned}$$

$$\sum_{s=\mathcal{T}}^{\infty} (e^{-c})^{s^{1-\alpha}} \leq \sum_{s=0}^{\infty} (e^{-c})^{s^{1-\alpha}} \leq \int_0^{\infty} e^{-cs^{1-\alpha}} ds = c^{\frac{\alpha}{1-\alpha}} \Gamma\left(\frac{1}{1-\alpha}\right) < \infty, \qquad 0 \leq \alpha < 1 \tag{4.3}$$

where $\Gamma$ is the Gamma function. It follows that the range of the speed of the flock is summable uniformly in time since it just depends on the initial velocities and the constant $\mathcal{T}$. So we proved the following result.

THEOREM 4. *Given the system* (4.2) *with weight functions according to Assumption 1, we have unconditional flocking for all* $0 \leq \alpha < 1$.

**4.2.1. The Cucker-Smale flocking.** An important remark is the similarity with the Cucker-Smale model for flocking [1]. The main version of this model considers (4.2) with the explicit non-linear relation between the agents $i, j$,

$$a_{ij}(t) = \eta(x_i(t) - x_j(t)) = \frac{K}{(\sigma^2 + ||x_i(t) - x_j(t)||^2)^{\beta}}$$

The authors establish flocking results for any $K > 0, \sigma > 0, \beta \geq 0$. The main result is that unconditional flocking can be achieved for $0 \leq \beta < 1/2$ which is similar to our result. For $\beta > 1/2$ however the flocking is not unconditional and additional assumptions on the initial conditions must be made. The analysis critically exploits the form and properties of $a_{ij}$.

**5. Discussion.** In this paper, we studied the consensus problem as a linear process on graphs. Literature suggests [12] that in case of uniform bounds there is a geometric rate of asymptotic convergence in the consensus subspace. Here, we approached the problem for the case that the edge weights are not uniformly bounded away from zero. We showed by a simple example that there are cases where, consensus is not guaranteed. We considered special vanishing weight functions with a rate exponent, $\alpha$, which the weights of the graph eventually dominate. The main result suggests that if $\alpha$ is within the interval $[0, 1]$, convergence results can be established, with rate slower than geometric.

Furthermore, we passed from the static and complete connectivity case to more general topologies, such as switching and random graphs, establishing relevant convergence results too. Towards this direction, the notion of the scrambling coefficient, $\gamma$ came up suppressing the interval of convergence to $[0, 1/\gamma]$.

Due to space limitations the continuous time case was omitted. Although the mathematical techniques differ, equivalent results are obtained. The full version of this work can be found in [6].

## REFERENCES

[1] Felipe Cucker and Steve Smale, *Emergent behavior in flocks*, IEEE Transactions on Automatic Control **52** (2007), no. 5, 852–862.
[2] Rick Durrett, *Probability. theory and examples, 4th ed.*, Cambridge, UK, 2010.
[3] Darald Hartfiel, *Nonhomogeneous matrix products*, World Scientific, 2002.
[4] Y Hatano and M Mesbahi, *Agreement over random networks*, IEEE Transactions on Automatic Control **50** (2005), no. 11, 1867–1872.
[5] Ilse C. F. Ipsen and Teresa M. Selee, *Ergodicity coefficients defined by vector norms*, SIAM J. Matrix Anal. Appl. **32** (2011), 153–200.
[6] C. Somarakis J. S. Baras, *Linear processes under vanishing communications - the consensus problem*, Tech. Report TR-2012-03, 2012.
[7] A Jadbabaie, J Lin, and A S Morse, *Coordination of groups of mobile autonomous agents using nearest neighbor rules*, IEEE Transactions on Automatic Control **48** (2003), no. 6, 988–1001.
[8] Ion Matei, Nuno Martins, and John S Baras, *Almost sure convergence to consensus in markovian random graphs*, 2008 47th IEEE Conference on Decision and Control (2008), 3535–3540.
[9] Mehran Mesbahi and Magnus Egerstedt, *Graph theoretic methods in multiagent networks*, Princeton University Press, 2010.
[10] Luc Moreau, *Stability of continuous-time distributed consensus algorithms*, 43rd IEEE Conference on Decision and Control (2004) (2004), 21.
[11] Maurizio Porfiri and Daniel L Stilwell, *Consensus seeking over random weighted directed graphs*, Automatic Control IEEE Transactions on **52** (2007), no. 9, 1767–1773.
[12] E Seneta, *Non-negative matrices and markov chains*, vol. 37, Springer, 2006.
[13] Alireza Tahbaz-Salehi and Ali Jadbabaie, *On consensus over random networks*, vol. 53, pp. 1315–1321, 2006.
[14] J Tsitsiklis, D Bertsekas, and M Athans, *Distributed asynchronous deterministic and stochastic gradient optimization algorithms*, IEEE Transactions on Automatic Control **31** (1986), no. 9, 803–812.