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Efficient and Robust Communication Topologies for Distributed Decision Making in Networked Systems

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***Abstract*— Distributed decision making in networked systems depends critically on the timely availability of critical fresh information. Performance of networked systems, from the perspective of achieving goals and objectives in a timely and efficient manner is constrained by their collaboration and communication structures and their interplay with the networked system's dynamics. In most cases achieving the system objectives requires many agent to agent communications. A reasonable measure for system robustness to communication topology change is the number of spanning trees in the graph abstraction of the communication system. We address the problem of network formation with robustness and connectivity constraints. Solutions to this problem have also applications in trust and the relationship of trust to control. We show that the general combinatorial problem can be relaxed to a convex optimization problem. We solve the special case of adding a shortcut to a given structure and provide insights for derivation of heuristics for the general case. We also analyze the small world effect in the context of abrupt increases in the number of spanning trees as a result of adding a few shortcuts to a base lattice in the Watts-Strogatz framework and thereby relate efficient topologies to small world and expander graphs.**

I. Introduction

The study of networked systems has gained a lot of interest in recent years. Many applications have emerged with the unifying theme being a group of agents achieving certain objectives via interaction in local levels. In most of these applications, existence of a central control unit which coordinates the agents actions is simply not possible. Also the agents are assumed to have some limited computational and processing power. Therefore, the objective has to be achieved through local interactions in a distributed manner.

Different performance measures can be defined for distributed algorithms, the most important of which are the speed of convergence, robustness to link/agent failures, and energy/communication efficiency. These performance measures cannot be achieved all at once and there is a trade off between the level at which the various measures can be fulfilled. All of these measures depend substantially on the structure of the network that the algorithm is running on as well as the dynamics of the system. In this paper, we address the problem of networks with robust structure in the presence of connectivity constraints by maximizing the number of possible spanning trees.

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In a cooperative system of autonomous agents, it is appropriate that the network is robust to link losses. The network formation is usually modelled sequentially. Local schemes - which use locally available information- have attracted much interest. Spanos and Murray [22] consider a localized notion of connectedness and study its relationship to the global connectivity of a network of vehicles. Zavlanos and Papas [28] address the problem of controlling the motion of a network of agents while preserving $k-$hop connectivity. Das and Mesbahi [8] have studied the problem of transmit power optimization with $k-$ node connectivity constraint in a wireless framework using semidefinite programming.

Meanwhile, it is also important to address the effect of adding links on the global measures of network robustness even in the cases where local computation of such global measures seem infeasible. The reason is twofold. First, it provides upper bounds on the improvement based on local schemes, determines fundamental limitations of the design, and provides benchmarks for comparison of local measures. Second, since network formation is a gradual process it is plausible that the nodes initially have some information about the network structure and use this information in the process of edge augmentation. Also, there exist methods (e.g. [2]) which provide nodes with information on the global topology based on local message passing algorithms. A valid question is then " given the present structure of the network and constraints on link establishment how should a node choose which link to establish in order to maximize a global measure of network robustness?" To answer this question, it is important to notice that autonomous agents are critically influenced by their understanding of the network topology. Therefore, their behavior and performance are functions of their initial knowledge or estimate of the group's topology.

Ghosh and Boyd [9] consider the optimization of the second smallest eigenvalue (also known as algebraic connectivity and the Fiedler eigenvalue) of the graph Laplacian as a measure of well-connectedness of the graph. They relaxed the combinatorial problem to a convex problem, used semidefinite programming to solve it, and provided a heuristic for large scale graphs. The Fiedler eigenvalue is a global measure of how fast local diffusion-type algorithms converge on a graph. It also provides a lower bound on the graph's edge and node connectivity. The number of spanning trees of a graph is a more general measure of graph connectivity. This number depends on the value of all the eigenvalues of the Laplacian matrix rather than only on the Fiedler eigenvalue and therefore is a more informative measure. As an example, in symmetric graphs such as rings

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the Fiedler eigenvalue has multiplicity of two or more. As a result its value, does not change with augmentation of an edge. However, adding an edge definitely changes the structure of a graph and its properties. This change is captured in the number of spanning trees of the graph.

In this paper we consider the number of spanning trees in the graph abstraction of the system as a global indicator of network robustness to link losses. Several measures have been proposed for characterizing the robustness of networks to link losses. Colbourn [7] provides a thorough literature survey on the combinatorics of network reliability. The classical approach in determining network reliability is to consider constant link loss probabilities for network edges and associate to each network configuration a polynomial, which determines the probability of connectedness of the corresponding configuration. We are interested in the case where all of the nodes send information that is intended to be used by all other agents, and therefore it is crucial to have a measure which considers reliable communication between all nodes (all to all). If the link loss probability is high, maximizing the number of spanning trees is essential for robustness of such systems [27], [25]. Kelmans [17], [18], [19] has most prominently studied the problem of graphs with the largest number of spanning trees. Tsen et al. [25] consider an algorithmic approach to finding the most vital edges for the number of spanning trees.

We address the problem of maximizing spanning trees with connectivity constraints. In Section II we provide the necessary background. In section III we state the problem and show that the general combinatorial problem can be relaxed to a convex optimization problem as in [9]. We also show that two issues of symmetrizing the graph and reducing the graph's effective resistance distance appear in the problem of maximizing the number of spanning trees; the optimal graph can be considered as a result of the interaction between these two factors. This is reminiscent of the logic behind the formation of small world graphs which is a trade-off between increasing clustering and decreasing distance. In section IV, we study the small world effect in the context of abrupt increases in the number of spanning trees as a result of adding shortcuts to a ring in the Watts-Strogatz framework. We use the analysis to provide insights for derivation of heuristics for the general case of optimal edge attachment. Due to space considerations some of the proofs are shortened and sketched. Detailed proofs can be found in [3].

II. Background

A. Motivation: Network Robustness and trust establishment

In networked systems of autonomous agents, a number of agents share and exchange information and collaborate to achieve a common objective. The information exchange is crucial to the performance of the system. Many metrics have been defined in different levels to address the reliable performance of communication networks. Examples of such measures are delay and throughput. At a higher level, it is important to address issues considering the topology of the network. The graph topological characteristics of a network provide fundamental limits on what it can achieve. In addition, in most realistic scenarios, it is important that the connectivity of agents is conserved as the environmental conditions or goals change in unexpected ways or the agents are confronted with adversarial agents or environments.

The number of spanning trees of a graph is a metric for well-connectedness of the graph [27], [7]. In a network with probabilistic link losses, the probability that there exists a path between any pair of nodes is equal to the probability of existence of a spanning tree. In classic reliability theory, a "reliability polynomial" is defined which determines how robust the network is to link losses. Consider a graph $\mathscr{G}$ with n nodes and e edges, a constant probability of link loss p, and let N_i denote the number of connected spanning subgraphs of the graph G. The reliability polynomial [7] is defined as:

$$Rel(\mathscr{G},p)=\sum_{i=n-1}^{e}N_i(1-p)^i p^{e-i}$$

Denote the number of spanning trees of graph G by $\tau(G)$. It can be verified [27] that for large p,

$$\tau(G)(1-p)^{n-1}p^{e-n+1}\leq Rel(G,p)\leq \tau(G)(1-p)^{n-1}$$

In network security, an important problem is to address the concept of trust and how it is established among agents based on previously observed or available evidence [4]. Trust establishment can be considered as a path problem on graphs. An agent i's assessment of trustworthiness of agent j can be calculated using the information contained in any path (relational or logical) from agent i to agent j. This problem has been addressed using a semiring method by Theodorakopoulos and Baras [24]. In such methods any spanning tree of the graph corresponds to a minimal graph which is necessary for all-to-all trust establishment. A larger number of spanning trees corresponds to a richer basis for trust establishment. Reference [20] uses a probabilistic model from reliability methods for trust assessments, in which the spanning trees are crucial.

B. Graph theory: Matrix-tree theorem and its variants

Consider a set of n agents and model the interconnection between them by a graph $\mathscr{G}=(\mathscr{V},\mathscr{E})$. The nodes of the graph, $\mathscr{V}=\{1,2,...,n\}$ represent the agents and the undirected edges $\mathscr{E}=\{l_1,l_2,...,l_e\}$ represent the links. The adjacency matrix for the graph, denoted by A is a symmetric n by n matrix with 1 in the ij^{th} position, if there is a link between nodes i and j. Given an arbitrary orientation of the edges of graph $\mathscr{G}$, the incidence matrix E of the graph is an n by e matrix, which has $1,-1$ or 0 in the ij^{th} position if the edge j is correspondingly an incoming edge to node i, an outgoing edge from node i, or not incident to node i. The degree of the i^{th} node, d_i equals the total number of edges incident to it.

Many graph invariants and parameters such as expansion parameters [14] and the number of spanning trees can be determined from the spectrum of the matrices related to graphs, most significantly the Laplacian matrix of the graphs. Consider D be a diagonal matrix whose i^{th} diagonal entry is equal to the degree of node i. The Laplacian of a graph is defined as $L = D - A = EE^T$ and is a positive semi-definite matrix: $\lambda_1 = 0 \leq \lambda_2 \leq ... \leq \lambda_n$. The eigenvalue $\lambda_1 = 0$ corresponds to the eigenvector $v_1 = \mathbf{1} = [11...1]^T$. If a graph is connected, $\lambda_2 > 0$ is known as the Fiedler eigenvalue of the graph.

The normalized Laplacian of a graph, $\mathscr{L}$ is defined as:

$$\mathscr{L} = D^{-1/2} L D^{-1/2}.$$

The normalized Laplacian is closely related to the stochastic transition matrix of the natural random walk on a graph,

$$P = D^{-1}A = I - D^{-1}L.$$

Using the similarity transformation, $D^{1/2}PD^{-1/2}$, it can be verified that $\lambda_i(P) = 1 - \lambda_{n+1-i}(\mathscr{L})$, where $\lambda_i(.)$ denotes the i^{th} smallest eigenvalue.

The number of spanning trees in a graph can be determined by Kirchhof's *matrix-tree theorem* [11]. Since for connected graphs L is a positive semi-definite matrix with $\lambda_2(L) > 0$, the nullspace of L is spanned by $\mathbf{1}$. On the other hand

$$L.Adj(L) = det(L).I_n = 0_n,$$

and L is symmetric; therefore $Adj(L)$ is a constant multiple of $J = \mathbf{1}\mathbf{1}^T$. This constant is equal to the number of the spanning trees of the graph as indicated by the Matrix-tree theorem [11].

Theorem 2.1: Let $\tau(\mathscr{G})$ denote the number of spanning trees in a graph G, $L, \mathscr{L}, P$ denote the Laplacian, normalized Laplacian, and natural random walk matrices of G, $Q = I - P$, and Q_i denote the i^{th} principal sub-matrix of Q, i.e. the matrix obtained by deleting the i^{th} row and column of Q, then:

$$Adj(L) = \tau(\mathscr{G})L. \tag{1}$$

$$\tau(\mathscr{G}) = \frac{1}{n}\prod_{j=2}^{n}\lambda_j(L). \tag{2}$$

$$\tau(\mathscr{G}) = \frac{1}{n}det(L + \frac{1}{n}J) \tag{3}$$

$$\tau(\mathscr{G}) = \prod_{j=2}^{n}\lambda_j(\mathscr{L})\frac{\prod_{i=1}^{n} d_i}{\sum_{i=1}^{n} d_i} \tag{4}$$

If $\mathscr{G}$ is connected,

$$\begin{aligned}\tau(\mathscr{G}) &= \prod_{j=1}^{n-1}(1-\lambda_j(P))\frac{\prod_{i=1}^{n} d_i}{\sum_{i=1}^{n} d_i} = \sum_{j=1}^{n} det(Q_j)\frac{\prod_{i=1}^{n} d_i}{\sum_{i=1}^{n} d_i} \\ &= det(Q_k)\prod_{i\neq k} d_i, \quad \forall k = 1, ..., n. \end{aligned}\tag{5}$$

Proof: (Sketch) The proofs of (1), (2), and (3) are classic. See [11]. The proof of (4) can be found in [6] and 5 is a direct result of (4). ■

Let $Z = (L + \frac{1}{n}J)^{-1}$. Consider an edge $l = (i, j)$. Since l is between nodes i and j, its incidence vector can be written as $f = e_i - e_j$, where e_i denotes a unit vector with 1 in the i^{th} entry. Therefore, we have $f^T Zf = z_{ii} - 2z_{ij} + z_{jj}$. This quantity is referred to as the *effective resistance* or the *resistance distance* between nodes i and j of the undirected graph $\mathscr{G}$ [10], [1]. If we consider the graph as a resistor network with 1Ω resistors on edges, this is the effective resistance when a voltage difference of $1V$ is applied across edge l.

III. Problem Statement

In this paper we are interested in the following problem: Given an initial graph topology, how should we add k edges, so that the resulting graph topology has the maximal number of spanning trees among all possible topologies. Consider a dynamic graph which evolves in time from a given topology $\mathscr{G}_0 = (\mathscr{V}_0, \mathscr{E}_0)$. Let's denote the complete graph on n vertices by $\mathscr{K}_n$. Also, denote the complement of a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ -which is the graph with the same vertex set but whose edge set consists of the edges not present in $\mathscr{G}$ - by $\bar{\mathscr{G}}$. So, $\mathscr{E}(\bar{\mathscr{G}}) = \mathscr{E}(\mathscr{K}_n) \setminus \mathscr{E}(\mathscr{G})$.

If we denote the operation of adding edge e to graph G by $Add(G, e)$, we consider the dynamic graph evolution:

$$\begin{array}{ll} G(t+1) = Add(G(t), u(t)), & t = 0, 1, ...k-1 \\ u(t) = e(t+1), & e(t+1) \in S \subseteq \mathscr{E}(\bar{G}(t)) \\ G(t) = \mathscr{G}_0 & \end{array}\tag{6}$$

The problem is to:

$$\begin{array}{rl} \text{maximize} & \tau(G(t+k)) \\ \text{subject to:} & (6) \end{array}\tag{7}$$

where $\tau(G(t)) = \prod_{i=2}^{n}\lambda_i(L)$.

This is a combinatorial optimization problem. If we denote the number of edges of $\mathscr{G}_0$ by e_0, there are $2^{\binom{n}{2}-e_0}$ possible edges, among which we should choose k. Even if we take a smaller candidate edge set S, the search space is very large and exhaustive search is not practical even for moderate graph sizes. In the sequel we will consider the convex relaxation of this problem. A similar approach has been independently pursued in [16]. We use the framework of [9].

A. General case and convex relaxation

Consider $G(1) = Add(\mathscr{G}_0, (l, p))$, then we can write:

$$L(1) = L(0) + (e_l - e_p)(e_l - e_p)^T,$$

where $L(i) \triangleq L(G(i))$. By indexing all candidate edges from 1 to m, denoting the corresponding incident vectors by $f_i = (e_{i_1} - e_{i_2})$ where i_1 and i_2 are the two ends of a candidate edge i, and introducing binary valued variables

$$x_i = \begin{cases} 1, & \text{if edge } i \text{ is chosen,} \\ 0, & \text{otherwise,} \end{cases}$$

we can write equation (7) as:

$$\begin{array}{rl} \text{maximize} & \tau(L(0) + \sum_{i=1}^{l} x_i f_i f_i^T) \\ \text{subject to:} & 1^T x = k \\ & x \in \{0,1\}^m. \end{array}\tag{8}$$

We now relax the above problem. Let $x > 0$. Consider $F_1(x) = [n\tau(L(x))]^{\frac{1}{n-1}} = (\prod_{i=2}^{n}\lambda_i)^{\frac{1}{n-1}}$ and $F_2(x) = \log\det(L(x)) = \log\prod_{i=2}^{n}\lambda_i$, which have the same maximizers.

Both of the above functions are concave functions of x. This is because, for example $g(L) = \det\log L$ is a concave function for its positive definite argument L, and $L(x) = L(0) + \frac{1}{n}J + \sum_{i=1}^{l} x_i f_i f_i^T$ is an affine function of x. Therefore the composition $g \circ L$ is a concave function. We can now solve the relaxed convex problem of maximizing $F_1(x)$ or $F_2(x)$ given the constraints. Considering the function $F_2(x)$, we have:

$$\begin{aligned} &\text{maximize} \quad F_2(x) = \log\det(L(x)) \\ &\text{subject to:} \quad 1^T x = k \\ &x > 0. \end{aligned} \tag{9}$$

The first order optimality condition requires that for maximum point $x^\star$,

$$\nabla F_2(x^\star)^T (x - x^\star) \leq 0$$

should hold for all $x > 0$ for which $1^T x = k$. Following [5], if $x^\star$ is a maximizer with a nonzero entry $x_i^\star$ and j is an arbitrary index with $j \neq i$, selection of x such that

$$x_p = \begin{cases} x_p^\star, & \text{if } p \neq i, j \\ 0, & \text{if } p = i \\ x_i^\star + x_j^\star & \text{if } p = j, \end{cases} \tag{10}$$

yields that at the maximum point $x^\star$ for all $j = 1, \ldots, m$,

$$\frac{\partial F_2(x^\star)}{\partial x_i} \geq \frac{\partial F_2(x^\star)}{\partial x_j},$$

Therefore at $x^\star$, $F_2(x)$ has equal derivative with respect to all positive x_i.

Taking the derivative yields that at the maximum, for all $x_i > 0$,

$$\begin{aligned} \text{Trace}\left((L_0 + \frac{1}{n}J + \sum_{i=1}^{m} x_i f_i f_i^T)^{-1} f_i f_i^T \right) &= f_i^T (L(x) + \frac{1}{n}J)^{-1} f_i \\ &= \lambda > 0. \end{aligned} \tag{11}$$

The term $f_i^T (L(x) + \frac{1}{n}J)^{-1} f_i$ is equal to the effective resistance (distance) between the two ends of the potential edge f_i. Since $F_2(x)$ is a concave function on a convex domain, the optimality conditions are also sufficient. Therefore, If feasible, one should add edges in a way that the effective resistance distance of all selected edges become equal. Also, the selected edges should be between the nodes with the highest resistance difference. Since it is not always possible to add the edges in this way, a good heuristic should make the difference between the effective resistance of the candidate edges as small as possible. We now address special cases of adding one or two edges, which provide more insight on how adding edges increases the number of spanning trees.

B. Adding one or two edges to a general graph

Consider adding an edge to a general initial graph, $G(0) = \mathscr{G}_0$, which results in a new graph, $G(1)$. As before enumerate the nodes of the graph from 1 to n. The following result holds.

Theorem 3.1: The optimal edge is between two nodes with maximal effective resistance distance.

Proof: Take two previously disconnected nodes $\alpha, \beta \in \{0, 1, \ldots, n\}$, and connect them by an edge. The incidence vector for this edge is $f = e_\alpha - e_\beta$. The number of spanning trees in $G(1)$ is:

$$\begin{aligned} \tau(G(1)) &= \frac{1}{n} \det\left(L + \frac{1}{n}J + (e_\alpha - e_\beta)(e_\alpha - e_\beta)' \right) \\ &= \left(1 + (e_\alpha - e_\beta)'(L + \frac{1}{n}J)^{-1}(e_\alpha - e_\beta) \right) \tau(\mathscr{G}_0) \end{aligned} \tag{12}$$

If we denote $Z = (L + \frac{1}{n})^{-1}$, then

$$\begin{aligned} \tau(G(1)) &= (1 + Z_{\alpha\alpha} - 2Z_{\alpha\beta} + Z_{\beta\beta}) \tau(\mathscr{G}_{(0)}) \\ &= \left(1 + R_{eff}(\alpha, \beta)\right) \tau(\mathscr{G}_{(0)}) \end{aligned} \tag{13}$$

Therefore, adding an edge between two nodes with the highest effective resistance results in the highest increase in the number of spanning trees of any general graph. ■

We now consider addition of two edges (α, β) and (γ, δ) to the initial graph $G^{(0)}$.

$$G(2) = Add\left(Add(\mathscr{G}_0, (\alpha, \beta)), (\gamma, \delta)\right).$$

The corresponding incidence vectors for the edges are, $f_{\alpha\beta} = e_\alpha - e_\beta$ and $f_{\gamma\delta} = e_\gamma - e_\delta$. Also, as before we let $Z = (L + \frac{1}{n}J)^{-1}$. Then,

$$\tau(G(2)) = \frac{1}{n} \det\left(L + \frac{1}{n}J + f_{\alpha\beta} f_{\alpha\beta}^T + f_{\gamma\delta} f_{\gamma\delta}^T \right) \tag{14}$$

Using the Sherman-Morrison-Woodbury formula for the inverse of a rank one modification of a matrix and some straight-forward calculations lead to:

$$\begin{aligned} \tau(G(2)) = \Big[&(1 + R_{eff}(\alpha, \beta))(1 + R_{eff}(\gamma, \delta)) \\ &- \left((z_{\gamma\alpha} - z_{\gamma\beta}) - (z_{\delta\alpha} - z_{\delta\beta}) \right)^2 \Big] \tau(\mathscr{G}_0) \end{aligned} \tag{15}$$

It can be seen that if the term $((z_{\gamma\alpha} - z_{\delta\alpha}) - (z_{\delta\alpha} - z_{\delta\beta}))^2$ were absent, the number of spanning trees would increase by a factor of $(1 + R_{eff}(\alpha, \beta))(1 + R_{eff}(\gamma, \delta))$. In that case it would suffice to join the two pairs of nodes with the highest effective resistance distance to maximize the number of spanning trees. However, this is not true in a general graph due to the interaction term $((z_{\gamma\alpha} - z_{\gamma\beta}) - (z_{\delta\alpha} - z_{\delta\beta}))^2$ in equation (15). Therefore, adding two edges (α, β) and (γ, δ) with the highest effective resistance distance, will result in the maximum spanning tree only in the symmetric cases where the nodes α and β are situated symmetrically with respect to nodes γ and δ. This is in line with the result of equation (11) which requires symmetry with regard to effective resistance distances.

The explicit formula for the cases of adding 3 or more edges can be derived in the same manner by using the Sherman-Morrison-Woodbury formula recursively. As the number of edges increases, more complex terms representing the interaction of the added edges appear in the formula. It is worthwhile to notice that two factors determine the optimal graph: minimizing a notion of distance and at the same time symmetrizing the graph. The resulting graph is the result of the interaction and trade-off between these two criteria. Such interaction and trade-off can be observed as the basic phenomenon in the formation of small world graphs, where the base graph provides necessary symmetry, while the shortcuts provide decrease in distance.

C. Special case: adding a shortcut to a ring

To illustrate Theorem 3.1 we consider adding a shortcut to a ring for which the explicit formula can be derived. Consider $\mathscr{G}_0$ to be a ring with the corresponding Laplacian matrix $L = D(0) - A(0)$ and natural random walk matrix $P_0 = (D_0)^{-1}(A_0)$. Take an arbitrary node. Without loss of generality we label this node as 1, and label the rest of the nodes as $2,3,...,n$ in a clockwise way. If $2 < j < n-1$, we refer to the potential edge $(1,j)$ as a shortcut. The length of such a shortcut is $j-1$. Let $\mathbb{G} = \{G^{(i)}\}_{i=3}^{n-1}$ denote a set of graphs where $G^{(i)}$ denotes the ring with an augmented shortcut between the nodes 1 and i. Denote the corresponding matrices by $\mathbb{L} = \{L^{(i)}\}_{i=3}^{n-1}$ and $\mathbb{P} = \{P^{(i)}\}_{i=3}^{n-1}$.

The number of spanning trees of a ring with n nodes is n. The problem is to find the graph $G^{(k)}$ for which $\tau(G^{(k)})$ is maximized. Since the node degrees are equal in all $G^{(i)}$, using equation (5), the term $\prod_{i \neq k} d_i$ is equal for all configurations and it suffices to maximize $\tau_1 = det(Q_1)$.

The following theorem characterizes the increase in the number of spanning trees as a function of shortcut lengths.

Theorem 3.2: In the problem of adding a shortcut to a ring, the number of spanning trees is an increasing function of the length of the shortcut. The maximum is attained by the graph $G^{(\frac{n+2}{2})}$ in case n is even. If n is odd the maximal value is attained by graphs $G^{(\frac{n+1}{2})}$ and $G^{(\frac{n+3}{2})}$.

Proof: (Sketch) The theorem can be proved directly by calculating $\tau_1(G^{(i)}) = \det(Q_1^{(i)})$, for $i = 3,...,n-2$. $Q_1^{(i)}$ is the first principal sub-matrix of $I - P^{(i)}$. The determinants can be calculated using the special structures of the matrices $Q_1^{(i)}$:

$$\det(Q_1^{(i)}) = \left[1 + \frac{2}{3n}\left((i - \frac{n+2}{2})^2 + \frac{n^2-2n+4}{4}\right)\right]\det(Q_1^{(0)})$$

Therefore the number of spanning trees is an increasing function of the shortcut length. Furthermore the maximum is attained by the graph $G^{(\frac{n+2}{2})}$ in case n is even. If n is odd the maximal value is attained by graphs $G^{(\frac{n+1}{2})}$ and $G^{(\frac{n+3}{2})}$. ■

IV. Small world effect and spanning trees

We consider the increase in the number of spanning trees in the Watts-Strogatz model for the small world effect. Watts and Strogatz [26] introduced a simple tunable model to explain the behavior of many real world complex networks. Their "Small World" model takes a regular ring or lattice and replaces the original edges by random ones with some probability $0 \leq \phi \leq 1$. Dynamical systems coupled in this way display enhanced signal propagation and global coordination, compared to regular lattices of the same size. At the same time small world graphs have reasonably high "clustering effect", which suggest that they are robust to link losses.

In the control community, several works have considered the small world effect as a measure of speed-up in the convergence of consensus problems [21], [23], [2], [15]. In our previous works, we developed a new method for investigating the effects of small world topologies by building on the probabilistic models of Higham [13], that established an equivalent representation of small world topologies as rare transitions among non-neighboring states in the Markov chain associated with a graph. We showed that since the small world model is obtained by stochastically adding or rewiring a few edges to a nominal graph, adding a small number of long distance edges is analogous to choosing graphs with low probability shortcuts and provided a probabilistic model based on our "mean field" model.

Here we adopt the same framework as in [13], [2] and study the increase of the number of spanning trees in the graph as a result of adding shortcuts with small weights. Consider the base lattice to have a ring topology on n nodes, $\mathscr{G}_0 = C(n,2)$, with adjacency and random walk matrices

$$\begin{aligned} A_0 &= Circ([0 \quad 1 \quad 0 \quad \dots \quad 0 \quad 1]) \\ P_0 &= Circ([0 \quad \frac{1}{2} \quad 0 \quad \dots \quad 0 \quad \frac{1}{2}]), \end{aligned} \tag{16}$$

where by $Circ(a)$ we mean a circulant matrix whose first row is given by the vector a. There are n spanning trees in a ring of size n. Instead of shortcuts with small probability, we assume applying negligible weights, ε to non-neighboring nodes. The resulting perturbed adjacency matrix is therefore:

$$A_\varepsilon = Circ([\varepsilon \quad 1 \quad \varepsilon \quad \dots \quad \varepsilon \quad 1]) \tag{17}$$

In the perturbed system, each node's degree is equal to the sum of the weights of the corresponding rows of the adjacency matrix, $2 + (n-2)\varepsilon$. Denote $D_\varepsilon = (2 + (n-2)\varepsilon)I$, the corresponding random walk matrix is equal to:

$$P_\varepsilon = D_\varepsilon^{-1} A_\varepsilon = \frac{1}{2 + (n-2)\varepsilon} A_\varepsilon. \tag{18}$$

The Laplacian and normalized Laplacian matrices (L_ε,$\mathscr{L}_\varepsilon$) can be defined similarly. P_ε can be written in terms of P_0, the random walk matrix of the unperturbed graph:

$$P_\varepsilon = \frac{2(1-\varepsilon)}{2 + (n-2)\varepsilon} P_0 + \frac{\varepsilon}{2 + (n-2)\varepsilon} J. \tag{19}$$

The following lemma determines the eigenvalues of P_ε in terms of those of P_0.

Lemma 4.1: The eigenvalues of P_ε are

$$\begin{aligned} &\lambda_n(P_\varepsilon) = 1, \\ &\lambda_{n-i}(P_\varepsilon) = \frac{2(1-\varepsilon)}{2 + (n-2)\varepsilon} \lambda_i(P_0), \quad i = 1,2,...,n-1. \end{aligned} \tag{20}$$

Proof: Consider the matrix $P_1 = \frac{2+(n-2)\varepsilon}{2(1-\varepsilon)} P_\varepsilon = P_0 + \frac{\varepsilon}{2(1-\varepsilon)} J$. Then,

$$\begin{aligned} \det(P_1 - \lambda I) &= \det(P_0 - \lambda I + \frac{\varepsilon}{2(1-\varepsilon)} \mathbf{1}\mathbf{1}^T) \\ &= [1 + \frac{\varepsilon}{2(1-\varepsilon)} \mathbf{1}^T (P_0 - \lambda I)^{-1} \mathbf{1}] \det(P_0 - \lambda I) \end{aligned} \tag{21}$$

Furthermore, for any $\lambda \notin Spec(P_0)$, $(P_0 - \lambda I)^{-1}\mathbf{1} = (1-\lambda)^{-1}\mathbf{1}$. Therefore for such λ,

$$\det(P_1 - \lambda I) = \left(1 + \frac{n\varepsilon}{2(1-\varepsilon)(1-\lambda)}\right)\det(P_0 - \lambda I) \quad (22)$$

It follows that the eigenvalues of P_1 are the same as the eigenvalues of P_0 except for $\lambda_n(P_1) = 1 + \frac{n\varepsilon}{2(1-\varepsilon)}$.

Since $P_\varepsilon = \frac{2(1-\varepsilon)}{2+(n-2)\varepsilon}P_1$,

$$\lambda_n(P_\varepsilon) = 1,$$
$$\lambda_i(P_\varepsilon) = \frac{2(1-\varepsilon)}{2+(n-2)\varepsilon}\lambda_i(P_0), \quad i = 1,2,...,n-1.$$

■

Now, we can state the following proposition:

Proposition 4.1: Let $\varepsilon = \frac{K}{n^\alpha}$, $\alpha > 1$, and consider the ratio $r = \frac{\tau(\mathscr{G}_\varepsilon)}{\tau(\mathscr{G}_0)}$, which measures the increase in the number of spanning trees as a result of adding ε weights:

- For $\alpha > 3$, the effect of the perturbation is negligible.
- $\alpha = 3$ is the onset of the effectiveness of shortcuts.
- For $1 < \alpha \leq 3$, the shortcuts dominantly increase the number of spanning trees, i.e. $\lim_{n\to\infty} \frac{\tau(\mathscr{G}_\varepsilon)}{\tau(\mathscr{G}_0)} = \infty$.

Proof: (Sketch) Using equation (5),

$$r(n) = \frac{\tau(\mathscr{G}_\varepsilon)}{\tau(\mathscr{G}_0)} = \left(\frac{\prod_{j=1}^{n-1}(1-\lambda_j(P_\varepsilon))}{\prod_{j=1}^{n-1}(1-\lambda_j(P_0))}\right)\cdot\left(\frac{\prod_{i=1}^{n} d_i(\mathscr{G}_\varepsilon)}{\prod_{i=1}^{n} d_i(\mathscr{G}_0)}\right) \cdot\left(\frac{\sum_{i=1}^{n} d_i(\mathscr{G}_0)}{\sum_{i=1}^{n} d_i(\mathscr{G}_\varepsilon)}\right) \quad (23)$$

The result follows by computing the three terms in the limit of $r(n)$ as $n \to \infty$. ■

As we have argued in [15], this probabilistic interpretation leads to a construction of small world networks by switching between graphs with low probability shortcuts. At each switching interval a few shortcuts are generated uniformly. Similarly, one can think of generating random spanning trees. In a recent paper Goyal et al. [12] have shown that the union of a few random spanning trees has constant edge expansion ratio and can be considered as *expander graphs.*

Expander graphs [14] capture the notion that any "local" set of nodes can access a large "global" neighborhood very efficiently. In a $d-$regular graph on n nodes $\mathscr{G} = (V,E)$, the edge expansion ratio is defined as: $h(G) = \min_{\{S||S|\leq\frac{n}{2}\}} \frac{\partial|S|}{|S|}$, where $|S|$ is a set of nodes and ∂S is the set of edges that separates S from its complement. Expander graphs are families of graphs, for which the expansion ratio is uniformly bounded away from zero as n increases. The expansion ratio of a graph is directly related to its spectrum by Cheeger's inequality. An algebraic limit for expansion is determined by Alon in terms of the spectral gap [14].

Efficient heuristics should consider symmetrizing the graph and adding edges between nodes with high resistance distance. Since the problem of adding one or two shortcuts is less complex, an efficient method is to solve such smaller problems, determine the best choices of edge augmentation for a set of nodes, and probabilistically switch between these configurations.

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