Proceedings of the
46th IEEE Conference on Decision and Control
New Orleans, LA, USA, Dec. 12-14, 2007

Combined Single-Path Routing and Flow Control in Many-User Region: A Case for Nash Efficiency

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***Abstract*— We consider the problem of combined single-path routing and flow control, which is nonconvex and NP-hard to solve exactly. We focus on the "many-user" region, i.e. large networks that have far more users than bottleneck links, which is close to real network scenarios. We first show that by allowing a proportionally small number of users to use multipath routing, while keeping the remaining majority using single-path routing, results in a solution that achieves multipath optimality. Therefore it is conceptually plausible that in the many-user region a local algorithm can achieve solutions arbitrarily close to the optimal solution. To show this is indeed correct, we focus on the solutions brought out by the simplest local algorithm, the Nash algorithm. We first examine a special type of network and show that the Nash equilibrium exists and the Nash algorithm always converges. It is then shown that the 'price of anarchy', that is the gap between the worst Nash equilibrium and the social optimum, is bounded when the number of users goes to infinity. For general networks, it is not known whether there exists a Nash equilibrium. We introduce the concept of approximate Nash equilibrium, show its existence, and prove that it will be arbitrary close to the social optimum when the number of users is sufficiently large.**

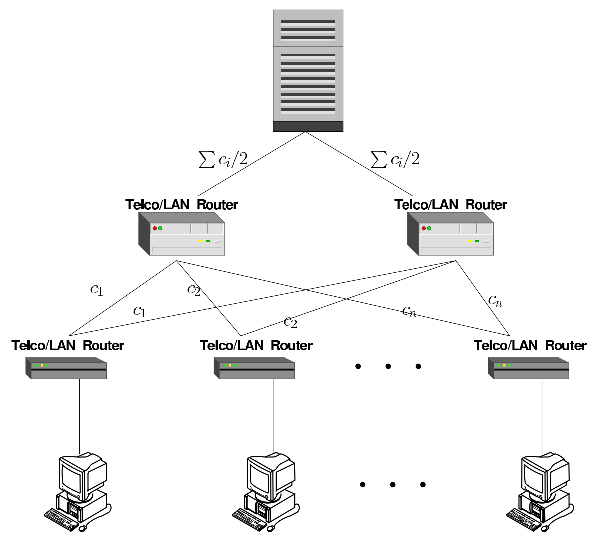

Fig. 1. A simple network which shows the problem of combined single-path routing and flow control problem is NP-hard [4].

I. Introduction

It is argued in [1] that a combined routing and flow control algorithm enables network users to improve their flow efficiency by transmitting packets over multiple fixed routes (multipath routing) simultaneously. However, within the current network infrastructure, although each user has more than one routes to send his traffic, he can only utilize one at each time. This motivates us to study the combined single-path routing and flow control problem.

Specifically, we use the network model from Kelly [2] in which there are N users and L bottleneck links. Each user i has a set of available paths m_i with cardinality M_i to send his traffic. The total number of all paths is $M = \sum_{l=1}^{N} M_l$. We index all the paths by the order of users so that $m_i = \{\sum_{l=1}^{i-1} M_l + 1, \cdots, \sum_{l=1}^{i} M_l\}$. We do not require the available paths for a single user to be disjoint from each other. We use the notation x_l, c_j, z_i to represent the flow rate of path l, the bandwidth of link j, and the flow rate of user i, respectively. We also use an $M \times L$, 0-1 matrix R as the routing matrix of the network to indicate the path/link relations. Distinct from the simultaneous routing and flow control problem, in single-path routing each user cannot simply take the aggregation of traffic flows of all his available paths. Instead at time t the effective user flow rate takes the following form $z_i(t) = \max_{l \in m_i} x_l(t)$. Thus, the network optimization problem for combined single-path routing and congestion control is

$$\begin{array}{c} \max_{x_l \geq 0} \sum_{i \in [N]} U_i(z_i) \\ \text{s.t. } z_i = \max_{l \in m_i} x_l, \forall i \in [N], \quad Rx \leq c. \end{array} \tag{1}$$

For comparison, the network optimization problem for combined multipath routing and congestion control is

$$\begin{array}{c} \max_{x_l \geq 0} \sum_{i \in [N]} U_i(z_i) \\ \text{s.t. } z_i = \sum_{l \in m_i} x_l, \forall i \in [N], \quad Rx \leq c. \end{array} \tag{2}$$

As we can see the only difference between these two problems is the effective user flow rate z_i. Using maximum, instead of sum over path rates, introduces nonconvexity into this problem, and consequently strong duality, which is fundamental for all distributed algorithms to solve the network optimization problem (see [2], [3]), does not hold.

This problem was first introduced by Wang, et al [4]. As the unsplittable flow problem, the problem of combined single-path routing and flow control is NP hard. As a result the stability conditions of price-based algorithms in specific networks [4], [5] are stringent. For expository purposes, we

Research partially supported by by the U.S. Army Research Laboratory under the CTA C&N Consortium Coop. Agreement DAAD19-01-2-0011, the National Aeronautics and Space Administration under Coop. Agreement NCC8235, and by the U.S. Army Research Office under MURI01 Grant No. DAAD19-01-1-0465 and under Grant No. DAAD19-02-1-0319.

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briefly describe a special type of network by which Wang, et al in [4] showed NP hardness. The network is shown in Fig. 1. There are $N+3$ nodes in the network: 1 server node, 2 intermediate router nodes, and N edge router nodes from which users can access the network. Each edge router i has two identical links with bandwidth c_i to each of the intermediate routers. Each intermediate router has one direct link to the server with bandwidth equal to half the sum of its incoming bandwidths from all the edge routers, that is, $\frac{1}{2}\sum_{i=1}^N c_i$. Assume each user wants to establish a link to the server and then each edge router has to decide which one of the two outgoing links it should select. It is straightforward to see that solving the network optimization (1) is equivalent to solving the number partitioning problem, that is to minimize $|\sum_{i\in S} c_i - \sum_{i\notin S} c_i|$ over all possible sets S. The latter is known to be NP-hard. Therefore the combined single-path routing and flow control problem is NP-hard.

Although the above result presents a somewhat pessimistic perspective for a complete algorithmic solution of the general problem (1), we can nonetheless proceed in the following directions which are still relevant to the real world scenario. First, since the exact optimal solution may be difficult to obtain, users may be content with a good approximation close to the true optimum. Second, the above NP-hardness proof relies on the assumption that the network size, i.e. the number of links, grows at a similar speed as the number of users. In reality there are many more users than links. So it is not too far from reality to consider the number of users growing while keeping the number of links fixed. In the case of the network in Fig. 1 this is similar in spirit to the scenario considered by Mertens [6] in his treatment of the number partitioning problem. He showed that when the ratio of the number resolution to the problem size is below a certain threshold, the number partitioning problem becomes easy to solve. Third, an important concern is whether the algorithm is local or not and how the algorithm uses local information. This is because improper use of local information, for example path selection decisions purely based on aggregate link prices, generally leads to route instability.

Based on the above three observations, we will focus on a simplistic local algorithm – Nash dynamics – and see how it performs when the number of users grows large. The intuition that there exists a local algorithm that works for the single-path routing problem as for the multipath routing problem is based on our result, obtained in Section II, that the former problem becomes "closer" to the latter one when the network size grows large. For a concrete example, using the same network as in Fig. 1, we will show in Section III route stability and bounded price of anarchy. That is, the gap between the result obtained by simple Nash dynamics and the global optimal is small on average. Our main result is derived in Section IV, where we show that in general networks all Nash equilibrium solutions are close to the optimal solution, in some sense, when the number of users is sufficiently large. We offer our conclusions in Section V.

II. From Single-Path to Multi-path Routing

First we will motivate our intuition by showing the relation between the combined single-path routing and flow control problem (1) and the multipath routing and flow control problem (2). Recall the definition of the conjugate function $f^*: X^* \to \mathbb{R}$ of $f: X \to \mathbb{R}$ as $f^*(y) = \inf_{x\in X}\{\langle x,y\rangle - f(x)\}$, where X and X^* is a pair of dual vector spaces defined by a bilinear operator $\langle\cdot,\cdot\rangle$. Then the bipolar function f^{**} of f is the conjugate of the conjugate of f, that is, $f^{**}(x) = \inf_{y\in X^*}\{\langle x,y\rangle - f^*(y)\}$. The following theorem by Falk states that the dual of a nonconvex optimization with linear constraints is equivalent to the "convex envelope" of the primal optimization,

Theorem 1 ([7]): For a compact set X, $f: X\to\mathbb{R}$ a lower semicontinuous function over X, consider the optimization problem

$$(P) \quad \max f(x), \text{ subject to } Ax\le b, x\in X,$$

with its dual problem

$$(P^*) \quad \min_{y\ge 0}\left(\max_{x\in X} f(x) - y^TAx + y^Tb\right).$$

The dual problem P^* is also the dual problem associated with the following problem

$$(P') \quad \max f^{**}(x), \text{ subject to } Ax\le b, x\in X.$$

Here f^{**} is the bipolar function of f. Further, if the Slater condition is satisfied, then the strong duality between P' and P^* holds: i.e. the maximum value of P' is equal to the minimum value of P^*.

The above theorem can be applied immediately to the combined single-path and flow control problem (1). The only thing left to be calculated is the bipolar function of $f_i(x_1,\cdots,x_{M_i}) \triangleq U_i(\max\{x_1,\cdots,x_{M_i}\})$. We may take the set X (and so X^*) in Theorem 1 as the one-point compactification of the positive orthant. We have

$$\begin{aligned}
& f_i^*(y_1,\cdots,y_{M_i}) \\
= \;& \inf_{x_1,\cdots,x_{M_i}}\left\{\textstyle\sum_{j=1}^{M_i} x_jy_j - U_i(\max\{x_1,\cdots,x_{M_i}\})\right\} \\
= \;& \inf_x\{x\min\{y_1,\cdots,y_{M_i}\} - U_i(x)\} \\
= \;& U_i^*(\min\{y_1,\cdots,y_{M_i}\}),
\end{aligned}$$

and

$$\begin{aligned}
& f_i^{**}(x_1,\cdots,x_{M_i}) \\
= \;& \inf_{y_1,\cdots,y_{M_i}}\left\{\textstyle\sum_{j=1}^{M_i} x_jy_j - U_i^*(\min\{y_1,\cdots,y_{M_i}\})\right\} \\
= \;& \inf_y\left\{\textstyle\sum_{j=1}^{M_i} x_jy - U_i^*(y)\right\} = U_i\left(\textstyle\sum_{j=1}^{M_i} x_j\right).
\end{aligned}$$

The above derivation uses the assumption that $U_i(\cdot)$ is a concave increasing function. Notice this utility function is exactly the utility function used in the combined multipath routing and flow control problem (2). Therefore we conclude,

Proposition 1: The dual problem of the combined single-path routing and flow control optimization (1) is equivalent to the combined multipath routing and flow control optimization (2). Therefore the duality gap is nonzero if and only if the optimal value of (2) is strictly larger than that of (1).

So the duality gap of the nonconvex optimization problem (1) can be interpreted as the "social" inefficiency caused by restricting every player to use only one path to route his traffic. It is then interesting to see what is the minimum relaxation of this restriction one should make in order to eliminate this gap. We will show that one only needs to make negligible modifications of this single-path rule to achieve multipath optimality in the many-player region. The derivation relies on a theorem by Shapley and Folkman (see [8] Appendix I) whose statement is as follows,

Theorem 2 (Shapley-Folkman): Given a finite family of sets $X_i \subset \mathbb{R}^m$, $i \in I$, for any $x \in \text{co} \sum_{i \in I} X_i$, there exists a subset $I(x) \subset I$, whose cardinality $|I(x)| \leq m$, such that $x \in \text{co} \sum_{i \in I(x)} X_i + \sum_{i \in I \setminus I(x)} X_i$.

Intuitively, the Shapley-Folkman Theorem says that the sum of a large number of nonconvex sets in a finite dimensional space is close to a convex set. In our context the set of achievable path rates for each user with single-path routing is nonconvex, while by the "smoothing" effect of the Shapley-Folkman Theorem the aggregate set of achievable path rates of all users is almost convex and close to its convex envelope, which is the set that corresponds to its multipath routing counterpart. In fact, we can establish the following proposition,

Proposition 2: The combined single-path routing and flow control problem (1) can achieve the same efficiency as the combined multipath routing and flow control problem (2) by allowing at most $L+1$ users to use multipath routing to transmit their traffic, where L is the number of bottleneck links in the network.

Proof: The idea follows from the calculation of duality gap in [8] Appendix I. First let us introduce an indicator function $\chi_l : \mathbb{R} \to \{0, -\infty\}$ for each link l such that

$$\chi_l(y) = \begin{cases} 0, & y \geq -c_l, \\ -\infty, & y < -c_l. \end{cases}$$

We will consider the following perturbed function

$$\begin{aligned} & \Phi(x_1, \cdots, x_M; d_1, \cdots, d_L) \\ \triangleq \; & \textstyle\sum_{i=1}^N U_i(\max_{j \in m_i} x_j) + \sum_{l=1}^L \chi_l \left(d_l - \sum_{j=1}^M R_{lj} x_j \right), \end{aligned}$$

and the perturbed maximization problem

$$V(d_1, \cdots, d_L) = \max_{\{x_i\} \in \mathbb{R}_+^M} \Phi(x_1, \cdots, x_M; d_1, \cdots, d_L).$$

It is clear that the optimal value of the combined single-path routing and flow control optimization (1) is the same as $V(0, \cdots, 0)$. Similarly, the bipolar $V^{**}(0, \cdots, 0)$ is equal to the optimal value of the corresponding multipath problem (2) by Proposition 1. We are then able to show $(0, \cdots, 0, V^{**}(0, \cdots, 0))$ is in the convex hull of the sum of $N+L$ sets, each of which corresponds to the rate distribution of a user or a link and his achievable utility. By the Shapley-Folkman Theorem 2, $(0, \cdots, 0, V^{**}(0, \cdots, 0))$ is in the sum of these $N+L$ sets, among which only at most $L+1$ sets need to be convexified. This is equivalent to the statement of the proposition. For details we refer to [9] Section 6.3. ■

Hence in the many-user region, the percentage of users that needs to be changed in order to transform the hard problem (1) into the easy problem (2) is vanishingly small given the fixed number of links. Also we can now see that the reason for the problem (1) being difficult to solve for the network in Fig. 1 is that there are at least as many bottleneck links as users. Therefore, intuitively, in the many-user region, the problem of combined single-path routing and flow control becomes close to its multipath counterpart and thus easier to solve. However we should first demonstrate in the next section that even in the network of Fig. 1, with as many bottleneck links as network users, a simple local algorithm leads to an asymptotically efficient result.

III. The Price of Anarchy - A Case Study

We consider the following type of noncooperative routing game in its normal form representation $([N], \{m_i\}, \{V_i\})$. The set of players $[N] = \{1, \cdots, N\}$ coincides with the set of users in the combined single-path routing and flow control optimization problem (1). Each player/user i has a finite number of strategies - its available routes - m_i. A pure strategy profile is then represented by an N-tuple $\sigma = (\sigma_1, \cdots, \sigma_N)$ where $\sigma_i \in m_i$ is the strategy chosen by player i. The set of pure strategy profiles is denoted by Σ. Player i's payoff function $V_i : \Sigma \to \mathbb{R}$ is defined by $V_i(\sigma) = U_i(z_i(\sigma))$ where $z_i(\sigma)$ is the optimal rate of player i in the following network optimal flow problem,

$$\begin{aligned} & \max_{x_l \geq 0} \textstyle\sum_{i \in [N]} U_i(z_i) \\ \text{s.t. } & z_i = x_{\sigma_i}, \quad \sigma_i \in m_i, \forall i \in [N], \quad Rx \leq c. \end{aligned} \tag{3}$$

Since the strategy set Σ is finite, there exists a pure strategy profile such that the resulting aggregate payoff in the above game achieves the maximum among all the possible strategies. This particular routing strategy is one of the Pareto optima of the game and along with the associated optimal flow rates, they are exactly the optimal solution of problem (1). To arrive at this Pareto optimum requires global coordination among players in general. An alternative way is to look at a solution concept of the game in which only local interactions are needed. A simple and also most well-known such solution is the Nash equilibrium of the game, which is defined as a strategy σ^{NE} such that

$$V_i(\sigma^{NE}) \geq V_i(\sigma_i, \sigma_{-i}^{NE}), \quad \forall \sigma_i \in m_i \text{ and } i \in [N].$$

Here σ_{-i}^{NE} denotes the $(N-1)$-tuple $(\sigma_1^{NE}, \cdots, \sigma_{i-1}^{NE}, \sigma_{i+1}^{NE}, \cdots, \sigma_N^{NE})$. We denote the set of Nash equilibria of the routing game by Σ^{NE}. The route update procedure to reach the Nash equilibrium can be described as follows. At each discrete time step t only a randomly selected player $p(t)$ switches its route to achieve a better resulting flow rate after the flow control mechanism is stabilized. This is known as Nash dynamics. Since this is a finite game, it is well known that there may not exist a pure Nash equilibrium in general, and if that is the case, the prescribed route update procedure will never terminate. We show below that in our routing game for the special network in Fig. 1, this routing instability will never occur.

Proposition 3: The routing game $(N, \{m_i\}, \{V_i\})$ for the network in Fig. 1 has pure Nash equilibria and consequently every Nash dynamics of the network terminates after a finite number of steps.

Proof: Let us define the set of strategy profiles $\Sigma = \{0,1\}^N$ where 0 represents the left route and 1 represents the right route. Recall that we use the notations $\sigma_i(t)$ and $z_i(t)$ for the route selection and actual bandwidth assigned to client i at the step t, respectively. Also define $P^0(t) = \{i \in [N] : \sigma_i(t-) = 0\}$ and $P^1(t) = \{i \in [N] : \sigma_i(t-) = 1\}$. Since every Nash dynamics can be decomposed into "rounds", during which the chosen players select the same route, let us denote by T_k the set of time steps spent at round k. It can be shown that $c_{p(t_k)}$ is strictly decreasing in k, where $t_k = \max T_k$. Then the stability result follows. For details we refer to [9] Section 6.2. ∎

Therefore in contrast to the NP-hardness of the "social optimum solution" for the network in Fig. 1, we have shown above that there always exists a Nash equilibrium, which can be reached in finite time by a simple local algorithm. It is natural to ask how far the Nash solution is from the network optimum. So next we consider the problem of evaluating "the price of anarchy" of this routing game, which measures the gap of the aggregate utility between the worst case Nash equilibrium and the social optimum. Our routing game is related to the line of research on the selfish routing problem of unsplittable flows (see [10], [11]), in which the bounded price of anarchy is shown if edge latency functions are polynomials of bounded degree. The main difference of our problem is that the traffic demands are elastic so that each player's payoff is not an explicit function of strategies. We adopt the definition of the price of anarchy as the *difference*, rather than the ratio as in most of the literature, of the aggregated utility function of the worst case Nash equilibrium to the global optimal value. We will start with the case when each user has a logarithmic utility function, which corresponds to proportional fairness allocation of network resources [2].

For N players in the network of Fig. 1, without loss of generality we assume that $0 < c_1 \le c_2 \le \cdots \le c_N$. Every strategy profile σ corresponds to a routing matrix $R(\sigma)$ and the rate allocations of the players are the solution to the following optimization problem,

$$\max_{z_i \ge 0} \textstyle\sum_{i=1}^N \log z_i, \quad \text{s.t.} \quad R(\sigma) z \le c,$$

where c is an appropriate column vector of link bandwidths.

It is easy to show that the solution satisfies the following property.

1) If $\sum_{i \in P^0} c_i \le \bar{c}$, $z_i = c_i$, $\forall i \in P^0$. Same applies to P^1.
2) If $\sum_{i \in P^0} c_i > \bar{c}$,

$$z_i = \min\left\{ c_i, \max_{k<i} \frac{\bar{c} - \sum_{j \in P^0, j<k} c_j}{|\{j \in P^0, j \ge k\}|} \right\}, \tag{4}$$

$\forall i \in P^0$. Same applies to P^1.

Suppose we fix the optimal aggregated utility function as

$$\sum_{i=1}^N \log c_i = 0, \text{ or } \prod_{i=1}^N c_i = 1.$$

Here "optimum" includes the situation when multipath routing is allowed. Therefore, our problem becomes,

$$\begin{array}{c} \min_{\sigma \in \Sigma^{NE}} J_0 = \max_z \sum_{i=1}^N \log z_i, \\ \text{s.t.} \quad R(\sigma) z \le c, \quad \prod_{i=1}^N c_i = 1. \end{array} \tag{5}$$

Recall Σ^{NE} is the set of pure Nash equilibrium profiles.

Proposition 4: The optimization problem (5) achieves its lower bound -1 when $N \to \infty$.

Proof: For any Nash equilibrium profile σ define $\{p_1^0, \cdots, p_{N_1}^0\} \triangleq P^0$ and $\{p_1^1, \cdots, p_{N_2}^1\} \triangleq P^1$ where $N_1 + N_2 = N$. The players in each set are ordered such that $c_1^0 \le \cdots \le c_{N_1}^0$ and $c_1^1 \le \cdots \le c_{N_2}^1$. Without loss of generality assume $\sum_{i=1}^{N_2} c_i^1 > \bar{c}$.

There exists an integer k_2, $0 \le k_2 < N_2$, such that there are k_2 players in P^1 whose allocated bandwidths are equal to their maximally possible bandwidths. It is straightforward to see that such k_2 players are the ones with the smallest bandwidths in P^1. Define $s_1^1 \triangleq \sum_{i=1}^{k_2} c_i^1$ and $s_2^1 \triangleq \sum_{i=k_2+1}^{N_2} c_i^1$. Again from the rule (4) and along with the inequality $s_1^1 \le k_2 c_{k_2}^1$ by definition, we have, $s_1^1 N_2 / k_2 \le \bar{c}$, So there exists $\delta_1 \ge 0$ such that

$$s_1^1 = \frac{k_2}{N_2} \bar{c} - \delta_1. \tag{6}$$

From some manupilations we can also get for some $\delta_2 \ge 0$,

$$s_2^1 = \frac{N_2 - k_2 + 1}{N_2} \bar{c} + \frac{1}{N_2 - k_2} \delta_1 - \delta_2 \tag{7}$$

We return back to the optimization problem (5). It is clear that the final rates z_i from the current strategy profile σ satisfy

$$z_i = \begin{cases} c_i, & p_i \notin \{p_{k_2+1}^1, \cdots, p_{N_2}^1\}, \\ (\bar{c} - s_1^1)/(N_2 - k_2), & \text{otherwise.} \end{cases}$$

Therefore we have

$$\begin{aligned} \exp(J_0) &= \prod_{p_i \notin \{p_{k_2+1}^1, \cdots, p_{N_2}^1\}} c_i \left(\frac{\bar{c} - s_1^1}{N_2 - k_2} \right)^{N_2 - k_2} \\ &= \left(\prod_{i=k_2+1}^{N_2} c_i^1 \right)^{-1} \left(\frac{\frac{N_2 - k_2}{N_2} \bar{c} + \delta_1}{N_2 - k_2} \right)^{N_2 - k_2} \\ &\ge \left(\frac{\frac{N_2 - k_2}{N_2} \bar{c} + \delta_1}{s_2^1} \right)^{N_2 - k_2} \ge \left(\frac{N_2 - k_2}{N_2 - k_2 + 1} \right)^{N_2 - k_2}. \end{aligned}$$

The first inequality is due to the fact that the arithmetic average is greater than the geometric average with equality when all the summands are equal to each other. The second inequality becomes equality when $\delta_1 = \delta_2 = 0$. Since it always holds $((N+1)/N)^N \searrow e^{-1}$, as $N \to \infty$, the conclusion of the proposition holds. An example to show the bound is tight is as follows. Consider a situation in which there are $2N - 1$ players and among those N players on the right route have bandwidths $\lambda(N+1)/N^2$ and $N-1$ players on the left route have bandwidths λ/N, where λ is the appropriate scaling constant so that the bandwidth constraint holds. The bound is achieved when $N \to \infty$. ■

The above derivation can be applied directly to the case of α-fair utility [12] $-x^{-\alpha}$ with $\alpha > 0$. Specifically, the network optimization problem is

$$\min_{\sigma\in\Sigma^{NE}} J_0 = \max_z \textstyle\sum_{i=1}^N -z_i^{-\alpha}, \quad \text{s.t.} \quad R(\sigma)z \le c,$$

and we fix the optimal value $\sum_{i=1}^N -c_i^{-\alpha} = -1$. Then using the same approach as in the case of proportional fair utility, we can show that the optimal value of the problem converges to -1 when $N \to \infty$. Therefore the price of anarchy of the routing game with α-fair utility becomes arbitrarily small when we have sufficiently large number of users.

We can extend the network to one with more than two links to the server and we can show the existence of a pure Nash equilibrium as well as the stability of Nash dynamics. It is observed that the price of anarchy grows with the number of bottleneck links. For example, in the case of logarithmic utility function with fixed social optimal value 0, the price of anarchy is at least $M-1$ as oppose to 1 when there are M links to the server. For details we refer to [9].

IV. Nash Equilibrium and Optimality - Asymptotic Results

In this section our intention is to show that in some sense the argument that the local Nash algorithm achieves optimality asymptotically when the number of users becomes large as compared to the number of bottleneck links is valid for general networks. It is well known that the Nash equilibrium causes efficiency loss in an exchange economy as opposed to the optimal, where each player acts like price-taker. It is plausible that in large economies this price-taking behavior is justifiable, since each player's ability to influence the price formation and consequently his gain to deviate from his true demand is diminishing when the number of players becomes large. This limit behavior of Nash equilibrium has been studied extensively (see for example, [13], [14] and references therein) and many indicate the convergence to the optimal. Note that our network flow optimizatoin problem is a special type of pure exchange economy. Our result confirms the economic intuition that the Nash equilibrium of our routing game (3) converges to the optimal solution in the many players region. It is worthwhile to mention here that this convergent behavior of Nash equilibria in the many-players region depends on the specific game form, since it is known that given a reasonably large strategy set, Nash equilibria of pure exchange economy do not shrink to the competitive equilibrium even when the set of user types is finite and there are infinite number of users of each type in the limit [15].

Although the pure Nash equilibrium of $([N], \{m_i\}, \{V_i\})$ exists for the network in Fig. 1, or its extension, whether a pure Nash equilibrium of the routing game exists for a general network is an open problem. To circumvent this issue we introduce a more general ϵ-Nash equilibrium: a strategy $\sigma \in \Sigma$ is an ϵ-Nash equilibrium if and only if $V_i(s, \sigma_{-i}) \le V_i(\sigma) + \epsilon$ for all $s \in m_i$ and all $i \in [N]$. Recall the notation $\sigma_{-i} = (\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_N)$. We denote the set of ϵ-Nash equilibria by $\Sigma^{NE(\epsilon)}$. Clearly $\Sigma^{NE} = \Sigma^{NE(0)}$ and $\Sigma^{NE(\epsilon)} \subset \Sigma^{NE(\epsilon')}$ if $\epsilon \le \epsilon'$.

Now we construct a concrete example of a many-user network. Here we use the term "network" for the network topology along with users' characteristics, that is, their utility functions and their available routes. Denote by $\mathcal{U}$ the set of concave strictly increasing functions defined on $\mathbb{R}_+$ and by M the set of all possible routes in a given network topology. For simplicity we consider the following "type sequence" of users for the network topology. There is a finite set $[T]$ of types and each $t \in [T]$ corresponds to a utility/strategy set pair $(U_t(\cdot), m_t) \in \mathcal{U} \times 2^M$. Denote the Nth network by $\mathcal{N}_N$. It consists of $n_N \in [N, N+T)$ users, among whom there are $\lceil w_t N \rceil$ users of type t for each $t \in T$. Here $w_t > 0$ can be considered as the percentage of type t users in the entire population and we have $\sum_{t\in T} w_t = 1$. In addition, the bandwidth of each link l of $\mathcal{N}_N$ is equal to Nc_l. The type sequence method (or "replica economy") offers a simple model similar to the real world scenario and its use is popular as the first step towards the study of the limiting behavior of large number of users in economic theory (see for example the case of core equivalence [16]).

Next for type t users in the network $\mathcal{N}_N$ we introduce a M_t-dim vector $\{v_{t,\tau}^N\}$ in which $v_{t,\tau}^N$ represents the number of type t users choosing the τth route within m_t. Apparently $\sum_\tau v_{t,\tau}^N = \lceil w_t N \rceil$ and all these $v_{t,\tau}^N$ users will have the same rate allocation. Therefore each $\{v_{t,\tau}^N\}$ corresponds to an equivalent class of strategy profiles σ in the sense of rate distribution. For a fixed $\{v_{t,\tau}^N\}$ the combined single-path routing and flow control problem for the network $\mathcal{N}_N$ can be written as

$$\begin{array}{c} \max_{x\ge 0} \sum_{t\in[T]} \sum_{\tau=1}^{M_t} v_{t,\tau}^N U_t(x_{t,\tau}) \\ \text{s.t.} \ \sum_{t\in[T]} \sum_{\tau=1}^{M_t} R_{t,\tau}^l v_{t,\tau}^N x_{t,\tau} \le Nc_l, \quad \forall l \in [L]. \end{array}$$

Here R is the routing matrix of the network which is invariant in N. Define $\bar{v}_{t,\tau} \triangleq v_{t,\tau}^N/N$ to be the scaled down version of $v_{t,\tau}$. Further denote the finite set $\bar{\mathcal{V}}_N$ to be all the possible $\{\bar{v}_{t,\tau}\}$ in which each component can be expressed by $\bar{v}_{t,\tau} = v_{t,\tau}/N$ for some $v_{t,\tau}^N$ for all t and τ. Then the "scaled down" version of the above optimization problem can be rewritten as

$$\begin{array}{l} V(\{\bar{v}_{t,\tau}\}) \triangleq \max_{x\ge 0} \sum_{t\in[T]} \sum_{\tau=1}^{M_t} \bar{v}_{t,\tau} U_t(x_{t,\tau}) \\ \text{s.t.} \ \sum_{t\in[T]} \sum_{\tau=1}^{M_t} R_{t,\tau}^l \bar{v}_{t,\tau} x_{t,\tau} \le c_l, \quad \forall l \in [L]. \end{array} \tag{8}$$

Also we can write the multipath version of the problem in the following form,

$$\begin{array}{c} \max_{x\ge 0} \sum_{t\in[T]} w_t U_t(z_t) \\ \text{s.t.} \ \sum_{t\in[T]} \sum_{\tau=1}^{M_t} R_{t,\tau}^l w_t x_{t,\tau} \le c_l, \quad \forall l \in [L], \\ z_t = \sum_{\tau=1}^{M_t} x_{t,\tau}. \end{array} \tag{9}$$

We have the following proposition regarding the existence of an ϵ-Nash equilibrium, *Proposition 5:* For any $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that for any $N > N(\epsilon)$ there exists an ϵ-Nash equilibrium of the routing game of $\mathcal{N}_N$.

Proof: The proof follows the parametric dependency property of convex programming (see [17] Theorem 1.17). Denote the solution of the multipath version of the problem (9) to be z_t^* and $x_{t,\tau}^*$ (if there are multiple solutions, we just pick one of them) and the solution to its dual problem to be p_l^*. Define $\bar{v}_{t,\tau}^* \triangleq w_t x_{t,\tau}^*/z_t^*$ and the aggregate price along the route $q_{t,\tau}^* \triangleq \sum_l R_{t,\tau}^l p_l^*$. We will focus on the strategy profile $\{\bar{v}_{t,\tau}^{*N}\}$, which we define to be the closest element in $\bar{\mathcal{V}}_N$ to $\{\bar{v}_{t,\tau}^*\}$ in l_∞. We complete the proof by showing that for any $\epsilon > 0$, there exists an integer $N(\epsilon)$, such that for any $N > N(\epsilon)$, any strategy profile corresponding to $\{\bar{v}_{t,\tau}^{*N}\}$ is an ϵ-Nash equilibrium. For details see [9] Section 6.3. ■

We use the notation $\mathbb{R}_{++}^T$ for the open set of the strictly positive orthant in $\mathbb{R}^T$. We will need the following technical fact before studying the properties of the ϵ-Nash equilibrium of the routing game – for its proof see [9] Section 6.3.

Proposition 6: For any $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}_{++}^T$ such that for all N, the aggregate price $q_{t,\tau}^N(\{\bar{v}_{t,\tau}^N\})$ for any type t user with strategy τ satisfies $q_{t,\tau}^N(\{\bar{v}_{t,\tau}^N\}) \in K$ where $\{\bar{v}_{t,\tau}^N\}$ is any ϵ-Nash equilibrium distribution of $\mathcal{N}$.

Now we state the main result of this section,

Theorem 3: For any $\delta > 0$, there exists an $\epsilon(\delta) > 0$ such that for any $0 < \epsilon < \epsilon(\delta)$ there is an $N(\epsilon, \delta)$ and for all $N > N(\epsilon, \delta)$ any ϵ-Nash equilibrium utilities $U_{t,\tau}^N$s for $\mathcal{N}_N$ satisfy $\max_{t,\tau} |U_{t,\tau}^N - U_{t,\tau}^{N*}| \le \delta$. Here $U_{t,\tau}^{N*}$ is the optimal utility for the type t user using route τ.

Proof: It suffices to show that the argument $\max_{t,\tau} |U_{t,\tau}^N - U_{t,\tau}^*| < \delta$, where $U_{t,\tau}^*$ is the optimal utility of the multipath version (9), since $U_{t,\tau}^{N*}$ converges to $U_{t,\tau}^*$ when $N \to \infty$. Suppose the statement does not hold. That is, for any $\varepsilon > 0$, and any $N > 0$, there exists $n > N$ and $0 < \epsilon < \varepsilon$ such that $\max_{t,\tau} |U_{t,\tau}^n - U_{t,\tau}^*| > \delta$ for an ϵ-Nash equilibrium $\{\bar{v}_{t,\tau}^N\}$. Therefore we can have two infinite sequences $\{\epsilon_n\}$ and $\{N_n\}$, such that $\epsilon_n > \epsilon_{n+1}$ and $N_{n+1} > N_n$ for all n, $\epsilon_n \to 0$ and $N_n \to \infty$ as $n \to \infty$, $|U_{t,\tau}^{N_n} - U_{t,\tau}^*| > \delta$ where $U_{t,\tau}^{N_n}$ corresponds to an ϵ_n-Nash equilibrium $\{\bar{v}_{t,\tau}^{N_n}\}$ of $\mathcal{N}_{N_n}$ for a fixed (t, τ) (since we can always take subsequences due to the finiteness of t and τ). By Proposition 6, all aggregate prices $q_{t,\tau}^{N_n}$ of an ϵ_n-Nash equilibrium belong to a compact set K. Then by uniform continuity of a continuous function over a compact set, there exists an infinite sequence $\{\eta_n\}$ such that $\eta_n > \eta_{n+1} > 0$ for all n, $\eta_n \to 0$ as $n \to \infty$, and 1) $|q_{t,\tau}^{N_n} - q_{t,\tau'}^{N_n}| < \eta_n$ for all $\tau, \tau' \in m_t$ such that $\bar{v}_{t,\tau}^{N_n} > 0$ and $\bar{v}_{t,\tau'}^{N_n} > 0$, and 2) $q_{t,\tau}^{N_n} < q_{t,\tau'}^{N_n} - \eta_n$ for all $\tau, \tau' \in m_t$ such that $\bar{v}_{t,\tau}^{N_n} > 0$ and $\bar{v}_{t,\tau}^{N_n} = 0$. Since $\bar{v}_{t,\tau}^{N_n}$ also belongs to a compact set, we can assume by taking a subsequence if necessary that $\bar{v}_{t,\tau}^{N_n} \to \bar{v}_{t,\tau}$ as $n \to \infty$ for all t and τ. Therefore asymptotically we have the Nash equilibrium strategy $\bar{v}_{t,\tau}$ with 1) $q_{t,\tau} = q_{t,\tau'}$ for all $\tau, \tau' \in m_t$ such that $\bar{v}_{t,\tau} > 0$ and $\bar{v}_{t,\tau'} > 0$, and 2) $q_{t,\tau} \le q_{t,\tau'}$ for all $\tau, \tau' \in m_t$ such that $\bar{v}_{t,\tau} > 0$ and $\bar{v}_{t,\tau} = 0$. However, we still have $|U_{t,\tau} - U_{t,\tau}^*| > \delta$. This is in contradiction with the fact that only the optimal solution of the multipath problem (9) has this property with the aggregate prices. ■

The simplistic local routing algorithm, Nash dynamics, leads to the optimal solution of the combined single-path routing and flow control problem (1) when the number of users becomes sufficiently large in a general replica network.

V. Conclusions

Our focus on the combined single-path routing and flow control problem is mostly on the descriptive side. We have shown that approximate Nash equilibria are sufficiently close to the social optimum in the many-user region, although we have not provided a precise rate or bound of this convergence. It is intuitive from Proposition 2 that it will be helpful if we can exactly find those $L+1$ users who cause the difference between the multipath problem and the single-path problem. But since the Shapley-Folkman theorem is nonconstructive, it is difficult to go in that direction. It is unknown whether the computational complexity of the combined single-path routing and flow control problem for a fixed number of bottleneck links is still NP hard. These are topics for future study.

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