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Maximum Entropy Models, Dynamic Games, and Robust Output Feedback Control for Nonlinear Systems*

John S. Baras

Abstract— **In this paper, we develop a framework for designing controllers for general, partially observed discrete-time nonlinear systems which are robust with respect to uncertainties (disturbances). A general deterministic model for uncertainties is introduced, leading to a dynamic game formulation of the robust control problem. This problem is solved using an appropriate information state. We derive a partially observed nonlinear stochastic model as the maximum entropy stochastic model for the nonlinear system. A risk-sensitive stochastic control problem is formulated and solved for this partially observed stochastic model. The two problems are related using small noise limits. These small noise asymptotics are for the first time justified as the appropriate randomization, using time asymptotics of the Lagrange multipliers involved in the maximum entropy model construction. Thus for the first time a complete unification of deterministic and randomized uncertainty models is achieved. Various interpretations, consequences and applications of this unification are explained and discussed.**

I. INTRODUCTION

We revisit the robust output feedback control problem for general nonlinear systems of the type

$$\begin{cases} x_{k+1} &= f(x_k, u_k), \\ y_{k+1} &= h(x_k, u_k), \qquad k = 0, 1, \ldots, M-1. \end{cases} \tag{1}$$

Here, $x_k \in \mathbf{R}^n$ denotes the state of the system, and is not in general directly measurable; instead an output quantity $y_k \in \mathbf{R}^p$ is observed. The control input is $u_k \in U \subset \mathbf{R}^m$. The system behavior is determined by the functions $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, $h : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^p$. It is assumed that the origin is an equilibrium for the system (1): $f(0,0) = 0$, and $h(0,0) = 0$.

In many applications we are interested to analyze the behavior of the system (1) in the presence of *model uncertainties* or *disturbances*; e.g., as arising from modeling errors, sensor noise, parametric variations, etc.. In almost all applications *performance and operational robustness* is of paramount importance. By that we mean the design of system structures and input strategies that can sustain desired performance and operation despite model uncertainties and/or signal disturbances. Such problems are widely known as *robust control-communication-signal processing* problems. In this paper we refer collectively to such problems as *robust decision and control* problems, or simply *robust control* problems. We propose and solve a general robust control problem for nonlinear systems, and in the process we develop a new and deeper understanding of the fundamental principles that support the framework that has been developed for linear systems (e.g. [6], [12], [4], [11]), as well as nonlinear systems [2], [36], [37], [38], [39], [20]. We selected to present the discrete time case in this paper, due to space limitations and in order to make the exposition easier. Our constructions and results hold also for the continuous time case, albeit the technical development is much more complicated and lengthy; we refer the reader to [42]. The work and results reported here are a natural extension of our results on automata [41]. At the same time we obtain a generalization of our earlier results [8], in that we consider more general nonlinear systems here (of the same generality as we considered in [36]).

The starting point of our approach is motivated by the method developed in [6], [12], [3], [13], [7], [2], [11], [1], [4], [8], [36], [37], [38], [39], [20]. We then develop a general framework for robust output feedback control of nonlinear systems, by carefully studying two basic methodologies for representing model uncertainties and signal disturbances: a deterministic one and a stochastic one. We investigate the relationship between the two resulting design methodologies for robust output feedback control, and establish a certain "duality" between them. Key to this linkage is the formulation of the robust control problem as a dynamic game between two players: *nature* (who selects model uncertainties and signal disturbances) and the *control designer* (who selects control strategies). When we use deterministic models for the system uncertainties the corresponding game is a deterministic game, while when we use stochastic models for the system uncertainties the corresponding game is a stochastic one. The relationship between the two design methods is a consequence of the relationship between the deterministic and the stochastic games.

When we model system uncertainties stochastically, the nonlinear system (1) is transformed to a partially observed stochastic nonlinear system (POSNLS) or Hidden Markov Model (HMM) [5], or chapter 5 of [18]. HMMs [10], [18], [17] have been studied extensively and numerous filtering, estimation, and control problems for them have been proposed and employed in applications. As already stated these models use a *probabilistic* description of system uncertainties and signal disturbances. Over the last fifteen years the

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J. S. Baras is with the Institute for Systems Research and the Department of Electrical & Computer Engineering, University of Maryland, College Park, MD 20742, USA. {baras}@isr.umd.edu

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output robust control problem has been investigated for various classes of systems, within the context of the so-called H^{∞} control. This substantial body of research results established the equivalence of the output robust control problem (or "four-block problem"), with the problem of solving a non-cooperative two player deterministic dynamic game, as well as with the problem of solving a single player *risk-sensitive* and partially observed stochastic optimal control problem. The equivalence of these three problems has been established for various system and performance metric models, but it has principally been interpreted and understood as a means for obtaining the solution of any of these problems in terms of the solution of the other. A key conclusion from these earlier research results is that risk-sensitive controllers are very closely related to robust controllers, see [4], [8], [36], [37], [38], [39], [20]. This discovery created quite a research activity on these problems, and it broke with the previously typical use of *risk-neutral* stochastic optimal control formulations for the majority of applications involving probabilistic models of uncertainty. It was the early work of Jacobson [6], Whittle [13] and few others that made it clear at first, that a controller more conservative than the risk-neutral one could be very useful.

A key and differentiating contribution in our earlier work [8], [36], [37], [38] was a framework that incorporates a *separation principle*, which in essence permits the replacement of the original output feedback problem by an equivalent one with full information, albeit infinite dimensional. We also introduced the use of an *information state* for solving the partially observed dynamic games. Indeed the information state for the games was obtained as an asymptotic limit of the information state for the risk-sensitive stochastic control problem; the latter arising from general principles of stochastic control, see [10]. The culmination of our earlier results was that the resulting feedback controller has an observer/controller structure, where the *observer* is the dynamical system describing the evolution of the information state. Indeed the control law is a *memoryless* (or instantaneous in time) function of the information state.

Yet, despite these developments, from a deeper systems perspective, a key question that remained unanswered was the following. It is clear that the risk-sensitive stochastic control problem involved in these equivalences, represents a *randomization* of the robust output feedback control problem. It is also clear that it represents a particular randomization. As is true in many other problems this randomization reduces the computational complexity of the underlying computational problem for computing robust output feedback controls. In the present paper we investigate exactly this deeper question, following the methods and goals of our recent work for automata [41]. Namely, what is the deeper fundamental principle leading to the particular randomization used in the risk-sensitive stochastic control formulation of the robust control problem? The answer, established here, is that this randomization is in fact equivalent to the construction of a *maximum entropy model* [27], [28], [31], [34], [23], which is a carefully constructed HMM. In this paper we establish this result for discrete time general nonlinear systems (1) and the associated HMMs. Establishing the result for continuous time nonlinear systems and hybrid systems, will be done elsewhere [42].

We establish the result by first reviewing our earlier work [8], [36]. We formulate the robust output feedback control problem for (1) and summarize the description of its equivalence to a deterministic partially observed dynamical game. We then formulate the robust output feedback control problem for the maximum entropy randomization of (1), following ideas from [22], [40], [16], [43]. We then solve the *risk-sensitive* stochastic optimal control problem for the resulting continuous state, discrete time HMM. Our solution, which is interesting in itself, leads us to the solution of the robust output feedback control problem for nonlinear systems. Finally, we link the two problems in yet another way by employing large deviation limits as in [8]. As in [41] we obtain another generalization from earlier approaches by us and others, in that we use a substantial generalization of the *finite gain* condition [8], [36], [20], which allows us to treat general constrained robust control problems, *vs* just robust stabilization problems.

The robust output feedback control problem for (1) is formulated in Section II; this entails defining deterministic uncertainty (disturbance) mechanisms with associated cost functions. In Section III, a stochastic uncertainty (disturbance) model is derived via the principle of maximum entropy modeling [27], [28], [31], [34], [23]. In the same Section we also derive the duality of this randomization with a partially observed risk-sensitive control problem. The risk-sensitive control problem is solved, and the large deviation principle is invoked (i.e. a small noise limit is evaluated) and used in Section IV to solve the robust output feedback control problem of Section II.

II. Output Robust Control Problem with Deterministic Uncertainty Models

A. Deterministic Perturbation

We model the influence of disturbances as follows. We consider the discrete-time nonlinear system (plant) (1) Σ augmented with two additional (disturbance) inputs w and v:

$$\begin{cases} x_{k+1} &= b(x_k, u_k, w_k), \\ y_{k+1} &= c(x_k, u_k, v_k). \quad k = 0, 1, \ldots, M-1. \end{cases} \tag{2}$$

Here, $w_k \in W \subset \mathbf{R}^r$ and $v_k \in V \subset \mathbf{R}^s$ are the disturbance inputs. The functions $b : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^r \to \mathbf{R}^n$, $c : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^s \to \mathbf{R}^p$, are required to satisfy the following consistency conditions: $b(x, u, 0) = f(x, u)$ and $c(x, u, 0) = h(x, u)$ for all x and u. We have used the origin (0) in these conditions w.l.o.g., in that 0's [9] play the role of "zero inputs", so that when no disturbances are present (i.e. $w_k \equiv 0$, and $v_k \equiv 0$), the behavior of (2) is the same as (1). The set of possible initial states is denoted $N_0 \subset \mathbf{R}^n$, and assumed to contain 0, while the set of possible future states for the disturbance model (2) is

$$N_{\mathbf{X}}(x,u) = \{b(x,u,w) \,:\, w \in W\} \;\subset\; \mathbf{R}^n,$$

and the corresponding set of possible future outputs is

$$N_{\mathbf{Y}}(x,u) = \{c(x,u,v) \,:\, v \in V\} \;\subset \mathbf{R}^p.$$

These sets can be thought of as "neighborhoods" of the nominal future values $f(x,u)$, $h(x,u)$, and are determined by the maps b and c. These can be designed as best fitting the application at hand.

B. Cost Functions

To quantify the effects of the disturbances, a measure of their "sizes" is required. To this end, we specify functions

$$\phi_w : \mathbf{R}^r \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}, \qquad \phi_v : \mathbf{R}^s \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R},$$

with the following properties:

$$\begin{cases} \phi_w(0;x,u) = 0 & \text{for all } x \in \mathbf{R}^n, u \in \mathbf{R}^m, \\ \phi_w(w;x,u) \geq 0 & \text{for all } 0 \neq w \in W, x \in \mathbf{R}^n, u \in \mathbf{R}^m, \end{cases}$$

$$\begin{cases} \phi_v(0;x,u) = 0 & \text{for all } x \in \mathbf{R}^n, u \in \mathbf{R}^m, \\ \phi_v(v;x,u) \geq 0 & \text{for all } 0 \neq v \in V, x \in \mathbf{R}^n, u \in \mathbf{R}^m, \end{cases}$$

and

$$\begin{cases} \beta(0) = 0, \\ +\infty > \beta(x_0) \geq 0 \quad \text{for all } x_0 \neq 0 \in N_0, \\ \beta(x_0) = +\infty \quad \text{for all } x_0 \notin N_0, \end{cases}$$

We think of $\phi_w(w;x,u)$ as the 'magnitude' of the disturbance w as it affects the system when it is in state x with control u applied, and $\phi_v(v;x,u)$ as the 'magnitude' of the disturbance v when in state x with control u applied. The cost function β specifies the 'amount of uncertainty' regarding the initial state.

Associated with these cost functions are quantities which define the optimal one-step cost of transferring from x to x'' and the optimal cost of producing the output y''. These quantities will be used in the solution of the robust control problem below. They are defined by

$$\begin{aligned} U(x,x'';u) &\triangleq \inf_{w \in W} \{\phi_w(w;x,u) \,:\, x'' = b(x,u,w)\}, \\ V(x,y'';u) &\triangleq \inf_{v \in V} \{\phi_v(v;x,u) \,:\, y'' = c(x,u,v)\}. \end{aligned} \tag{3}$$

We adopt the convention that the minimum over an empty set equals $+\infty$. Thus U and V are extended real valued functions. Note that

$$\begin{aligned} U(x,f(x,u);u) &= 0 \quad \text{for all } x \in \mathbf{R}^n, u \in U, \\ U(x,b(x,u,w);u) &\geq 0 \quad \text{for all } 0 \neq w \in W, x, u, \\ U(x,x'';u) &= +\infty \quad \text{if } x'' \notin N_{\mathbf{X}}(x,u). \end{aligned}$$

and

$$\begin{aligned} V(x,g(x,u);u) &= 0 \quad \text{for all } x \in \mathbf{R}^n, u \in U \\ V(x,h(x,u,v);u) &\geq 0 \quad \text{for all } 0 \neq v \in V, x \in \mathbf{R}^n, \\ V(x,y'';u) &= +\infty \quad \text{if } y'' \notin N_{\mathbf{Y}}(x,u). \end{aligned}$$

C. Robust Control

As part of the output robust control problem specification, we define an additional output quantity

$$z_{k+1} \;=\; \zeta(x_k,u_k), \tag{4}$$

where z_k takes values in $\mathbf{R}^l$, and $\zeta : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^l$. In the so called "four-block" formulation of the robust output feedback control problem, z_k represents the *regulated variable*, while y_k represents the *measured* (observed) variables available to the controller. We assume that the origin is the null element such that $\zeta(0,0) = 0$. A cost function for this output is also specified, with the properties

$$\begin{cases} \phi_z(0) = 0, \\ \phi_z(z) \geq 0 \quad \text{for all } z \in \mathbf{R}^l. \end{cases}$$

The output quantity z and its associated cost function ϕ_z encode the performance objective of the problem at hand. To summarize, the complete system is described by the equations

$$\begin{cases} x_{k+1} = b(x_k,u_k,w_k), \\ z_{k+1} = \zeta(x_k,u_k), \\ y_{k+1} = c(x_k,u_k,v_k), \qquad k = 0,1,\ldots,M-1. \end{cases} \tag{5}$$

The state variable x_k is not measured directly, and so the controller must make use of information available in the output signal $y_{0,k}$; i.e., the controller must be an *output feedback* controller. We denote by $\mathcal{O}_{k,k'}$ the set of non–anticipating control policies defined on the interval $[k,k']$; i.e., those controls for which there exist functions $\bar{u}_j : \mathbf{R}^{p(j-k+1)} \to U$ such that $u_j = \bar{u}_j(y_{k+1,j})$ for each $j \in [k,k']$.

The finite-time *output feedback robust control problem* generalizing [38] is then: given $\gamma > 0$ and a finite time interval $[0,M]$ find an output feedback controller $u \in \mathcal{O}_{0,M-1}$ such that

$$\sum_{k=0}^{M-1} \phi_z(z_{k+1}) \;\leq\; \beta(x_0) + \gamma \sum_{k=0}^{M-1} \left(\phi_w(w_k;x_k,u_k) + \phi_v(v_k;x_k,u_k)\right) \tag{6}$$

for all $(w,v) \in W^M \times V^M$, $x_0 \in \mathbf{R}^n$.

D. Dynamic Game

The robust control problem formulated above can be recast as a dynamic game problem, see, e.g., [36], [37], [38]. The payoff function for the controller $u \in \mathcal{O}_{0,M-1}$ (player 1) and disturbances $(w, v, x_0) \in W^M \times V^M \times N_0$ (player 2) is given by

$$J^\gamma(u, w, v, x_0) \triangleq -\beta(x_0) + \sum_{k=0}^{M-1} \phi_z(z_{k+1}) - \gamma\left(\phi_w(w_k; x_k, u_k) + \phi_v(v_k; x_k, u_k)\right). \quad (7)$$

We consider the upper game for this payoff given the dynamics (2). Define

$$J^\gamma(u) \triangleq \sup_{(w,v)\in W^M \times V^M} \sup_{x_0 \in N_0} \{J^\gamma(u, w, v, x_0)\}.$$

The bound

$$0 \le J^\gamma(u) \le M \sup_{z \in R^s} \phi_z(z) \quad (8)$$

is readily verified. The dynamic game problem is to find an output feedback controller $u^* \in \mathcal{O}_{0,M-1}$ such that

$$J^\gamma(u^*) = \inf_{u \in \mathcal{O}_{0,M-1}} J^\gamma(u). \quad (9)$$

Then if

$$J^\gamma(u^*) = 0, \quad (10)$$

the robust control objective (6) is achieved.

We will solve this dynamic game problem in Section IV.

III. Output Robust Control Problem with Stochastic Uncertainty Models

A. Maximum Entropy Model Construction

We follow [16]. The second law of thermodynamics asserts that a physical system in equilibrium has maximal entropy among all states with the same energy. Translating this into a probabilistic language and replacing entropy by the more general relative entropy, we are led to the following question. Let $\mathcal{C}$ be a class of probability measures on some measurable space $(E, \mathcal{E})$ and μ a fixed reference measure on $(E, \mathcal{E})$. What then are the probability measures in $\mathcal{C}$ minimizing $D(.|\mu)$? The universal significance of such minimizers has been put forward by Jaynes [27], [28]: *maximum entropy principle*. These minimizers arise also in the celebrated Sanov's theorem [16], [43], which is of fundamental importance to large deviation theory (LDT). In the general setting just described the relative entropy can be defined by $D(\nu|\mu) = sup_{\mathcal{P}} D(\nu_{\mathcal{P}}|\mu_{\mathcal{P}})$, where the supremum extends over all finite $\mathcal{E}$-measurable partitions $\mathcal{P}$ and $\nu_{\mathcal{P}}$ stands for the restriction of ν to $\mathcal{P}$. Equivalently, $D(\nu|\mu) = \nu(log f)$ if ν is absolutely continuous with respect to μ with density f, and $D(\nu|\mu) = \infty$ otherwise. This definition shows that $D(.|\mu)$ is lower semicontinuous in the so-called $\tau - topology$ generated by the mappings $\nu \to \nu(A)$ with $A \in \mathcal{E}$. Consequently, a minimizer does exist whenever $\mathcal{C}$ is closed in this topology. If $\mathcal{C}$ is also convex, the minimizer is uniquely determined due to the strict convexity of $D(.|\mu)$, and is called the *I - projection* of μ on $\mathcal{C}$ [30].

For our purposes here it suffices to consider the most classical case when $\mathcal{C}$ is defined by an integral constraint (or constraints). To this end we will denote by $\nu(g)$ the integral of some bounded measurable function $g : E \to \mathbf{R}^d$ with respect to ν. We will then assume that $\mathcal{C} = \{\nu : \nu(g) = a\}$ for suitable $a \in \mathbf{R}^d$. In other words we are interested in the constrained variational problem

$$minD(\nu|\mu), \text{ over } \nu(g) = a. \quad (11)$$

In this case we use a convex Lagrange multiplier calculus as follows. For any bounded measurable function $f : E \to \mathbf{R}$ let $L(f) = log\ \mu(e^f)$ be the log-Laplace functional of μ. We then have the variational formula (from duality)

$$D(\nu|\mu) = sup_f[\nu(f) - L(f)], \quad (12)$$

meaning that $D(.|\mu)$ and L are convex conjugates (i.e. Legendre-Fenchel transforms in duality [44]) of each other; c.f. Lemma 6.2.13 of Dembo and Zeitouni [43]. Let

$$J_g(a) = inf_{\nu:\{\nu(g)=a\}} D(\nu|\mu) \quad (13)$$

be the entropy distance of $\{\nu : \nu(g) = a\}$ from μ. Applying convex analysis (specifically Fenchel duality) then we get

$$J_g(a) = sup_{\lambda \in \mathbf{R}^d}[\lambda \cdot a - L(\lambda \cdot g)], \quad (14)$$

i.e. J_g is a partial convex conjugate of L (or in other terms, the Cramér transform of the distribution $\mu \circ g^{-1}$ of g under μ; c.f. Varadhan, chapter 9 of [16]. Moreover, if g is nondegenerate (in the sense that $\mu \circ g^{-1}$ is not supported on a hyperplane), then J_g is differentiable on the interior $I_g = int\{J_g < \infty\}$ of its essential domain. As a result we have the celebrated *Gibbs-Jaynes principle*, see chapter 3 of [16]:

Theorem 3.1: **The Gibbs-Jaynes Principle**. For any nondegenerate $g : E \to \mathbf{R}^d$, $a \in I_g$ and $\lambda = \nabla J_g(a)$ the probability measure

$$\mu_\lambda(dx) = Z_\lambda^{-1} e^{\lambda \cdot g(x)} \mu(dx) \quad (15)$$

on $(E, \mathcal{E})$ is the unique minimizer of $D(.|\mu)$ on $\{\nu : \nu(g) = a\}$. Here $Z_\lambda = e^{L(\lambda \cdot g)}$ is the normalizing constant.

Several generalizations of this result have been obtained, see for example [29], [30].

In statistical mechanics, the measures μ_λ of the above form are called *Gibbs distributions*, and the above theorem justifies that these are indeed the equilibrium distributions of physical systems satisfying a finite number of conservation laws. In mathematical statistics, such classes of probability measures are called exponential families. For us it is important that they provide the *maximum entropy* models needed (and used) in our theory.

B. Random Perturbation

The random perturbation (disturbance model) developed below is a stochastic analog of the deterministic perturbation model introduced in Section II. It is based on the maximum entropy principle described in Section III-A.

Let

$$\sigma = \Big\{\Big((x_0,y_0),(x_1,y_1),\dots(x_M,y_M)\Big)\Big\} \in \mathbf{R}^{n(M+1)} \times \mathbf{R}^{p(M+1)},$$

and let, $P_1^M(\sigma)$ be the joint probability density function of the set of random variables

$$\Big\{\Big((X_0,Y_0),(X_1,Y_1),\dots(X_M,Y_M)\Big)\Big\}.$$

i.e. the joint finite dimensional probability density function (pdf) of the state and output trajectories over the entire observation interval $[0,M]$. We choose the statistics of the stochastic system model in the least biased way [27], [28] by picking the joint pdf ($P_1^M(\sigma)$) that generates the *maximum entropy* over the set of all possible joint pdfs, while at the same time satisfying the constraints imposed by the 'observed levels' of disturbances. Thus we want to maximize

$$-\mathbf{E}\log\left(P_1^M(\sigma)\right)$$

over all pdfs that generate specified levels of disturbances. As explained in Section III-A there are many ways to specify constraints expressing observed quantities of the process trajectories. For simplicity of exposition we chose here constraints on observations expressed through the average costs of the disturbances; i.e. expectations with respect to the process measure. So the pdf we pick must satisfy:

$$\begin{aligned} \mathbf{E}U(x_i,x_{i+1};u) &= \alpha_i \quad \text{for } 0 \le i \le M-1,\\ \mathbf{E}V(x_i,y_{i+1};u) &= \beta_i \quad \text{for } 0 \le i \le M-1,\\ \mathbf{E}\beta(x_0) &= \gamma. \end{aligned}$$

In addition, the probabilities $P_1^M(\sigma)$ must integrate to one. We thus have to maximize a strictly concave function on the simplex subject to linear constraints. The function achieves the global maximum at the only critical point it has. This is exactly the case we described in section III-A, and we obtain the solution via Theorem 3.1. Equations (12,13,14) provide the means to compute the solution and characterize its properties. Due to space limitations we omit the lengthy and tedious details.

The most important for us properties of the solution are the asymptotic behavior of the Lagrange multiplier vector, the λ in the expression of the Gibbs distribution (measure) of Theorem 3.1. We use Lagrange multipliers ($\{\lambda_i, i=0,\cdot,M-1\},\{\mu_i, i=0,\cdot,M-1\},\nu,\kappa$). An analysis of the solution allows us to establish that the multipliers ($\{\lambda_i\},\{\mu_i\},\nu$) are monotonic functions of the corresponding disturbance levels $\{\alpha_i\},\{\beta_i\},\gamma$; they decrease monotonically as the disturbanceslevels are raised. For instance using the results of section III-A as applied to $\{\lambda_i\}$ we can show that

$$\frac{d\lambda_i}{d\alpha_i} \le 0$$

Also, when $\alpha_i = \inf_{x_i,x_j}\{U(x_i,x_j;u)\}$, $\lambda_i=\infty$. Thus when the disturbance levels go down to zero, the Gibbs distribution we get as the maximum entropy solution, reduces to the unperturbed partially observed deterministic nonlinear system we started with.

When we choose the expected disturbance levels so that the Lagrange multipliers ($\{\lambda_i\},\{\mu_i\},\nu$) are all equal to $1/\varepsilon$, the maximum entropy model we derived (15) is a *controlled Hidden Markov Model*; discrete-time and continuous state. It consists of an $\mathbf{R}^n$ valued controlled Markov process x_k^ε together with a $\mathbf{R}^p$ valued output process y_k^ε. The *combined* (joint) *state and output process* is Markov and it is therefore described by the conditional pdfs of state transitions and state to output values. We have:

$$\mathbf{P}^u\left(x_{k+1}^\varepsilon = x'' \,|\, x_k = x, x_{0,k-1}, u_k = u, u_{0,k-1}\right) = A^\varepsilon(u)_{x,x''}, \tag{16}$$

and

$$\mathbf{P}^u\left(y_{k+1}^\varepsilon = y'' \,|\, x_k = x, u_k = u\right) = B^\varepsilon(u,x)_{y''}, \tag{17}$$

where,

$$\begin{aligned} A^\varepsilon(u)_{x,x''} &\triangleq \tfrac{1}{Z_{x,u}^{\varepsilon,U}}\exp\left(-\tfrac{1}{\varepsilon}U(x,x'';u)\right),\\ B^\varepsilon(u,x)_{y''} &\triangleq \tfrac{1}{Z_{x,u}^{\varepsilon,V}}\exp\left(-\tfrac{1}{\varepsilon}V(x,y'';u)\right), \end{aligned} \tag{18}$$

where the functions U and V are defined by (3), and $Z_{x,u}^{\varepsilon,U}$ and $Z_{x,u}^{\varepsilon,V}$ are appropriate normalizing constants. Similarly, the initial distribution is

$$\rho^\varepsilon(x_0) = \tfrac{1}{Z_{x_0}^\varepsilon}\exp\left(-\tfrac{1}{\varepsilon}\beta(x_0)\right).$$

Thus where $\mathbf{P}^u$ is the probability distribution on $\mathbf{R}^{n(M+1)} \times \mathbf{R}^{p(M+1)}$ defined by a control policy $u \in \mathcal{O}_{0,M-1}$:

$$\mathbf{P}^u(x_{0,M},y_{1,M}) = \Pi_{k=0}^{M-1} A^\varepsilon(u_k)_{x_k,x_{k+1}} B^\varepsilon(u_k,x_k)_{y_{k+1}}\rho^\varepsilon(x_0)$$

The probability distribution $\mathbf{P}^u$ is equivalent to a distribution $\mathbf{P}^\dagger$ under which $\{y_k^\varepsilon\}$ is iid uniformly on $\mathbf{R}^p$, independent of $\{x_k^\varepsilon\}$, and $\{x_k^\varepsilon\}$ is a controlled Markov process as above. Let

$$\frac{d\mathbf{P}^u}{d\mathbf{P}^\dagger}\Big|_{\mathcal{G}_k} = \lambda_k^\varepsilon$$

where $\mathcal{G}_k$ is the filtration generated by $(x_{0,k}^\varepsilon, y_{1,k}^\varepsilon)$.

C. Cost

The cost function is defined for admissible $u \in \mathcal{O}_{0,M-1}$ by

$$J^{\gamma,\varepsilon}(u) = \mathbf{E}^u\left[\exp\frac{1}{\gamma\varepsilon}\sum_{l=0}^{M-1}\phi_z\left(\zeta(x_l^\varepsilon,u_l)\right)\right] \tag{19}$$

and the *output feedback risk-sensitive stochastic control problem* for the HMM (18) is to find $u^* \in \mathcal{O}_{0,M-1}$ such that

$$J^{\gamma,\varepsilon}(u^*) \;=\; \inf_{u\in\mathcal{O}_{0,M-1}} J^{\gamma,\varepsilon}(u).$$

In terms of the reference measure, the cost can be expressed as

$$J^{\gamma,\varepsilon}(u) \;=\; \mathbf{E}^\dagger\left[\lambda^\varepsilon_M \exp\frac{1}{\gamma\varepsilon}\sum_{l=0}^{M-1}\phi_z\left(\zeta(x^\varepsilon_l, u_l)\right)\right]. \quad (20)$$

D. Information State

Following [8], [36], [37], [38], we define an information state process $\sigma^{\gamma,\varepsilon}_k \in \mathbf{R}^n$ by the relation

$$\sigma^{\gamma,\varepsilon}_k(x) \;=\; \mathbf{E}^\dagger\left[I_{\{x^\varepsilon_k = x\}}\exp\frac{1}{\gamma\varepsilon}\sum_{l=0}^{k-1}\phi_z\left(\zeta(x^\varepsilon_l,u_l)\right)\lambda^\varepsilon_k \mid \mathcal{Y}_k\right], \quad (21)$$

where $\mathcal{Y}_k$ is the filtration generated by the observation process $y^\varepsilon_{1,k}$, and $\sigma^{\gamma,\varepsilon}_0(x) = I_{\{x=x_\emptyset\}}$.

The evolution of this process is determined by an operator with kernel $\Sigma^{\gamma,\varepsilon}(u,y'')$ defined by

$$\Sigma^{\gamma,\varepsilon}(u,y'')_{x,x''} \;\stackrel{\triangle}{=}\; A^\varepsilon(u)_{x,x''}\,\Psi^\varepsilon(x,y'')\,\exp\frac{1}{\gamma\varepsilon}\phi_z\left(\zeta(x,u)\right). \quad (22)$$

Indeed, the information state is the solution of the recursion (c.f. [8], [38])

$$\left\{\begin{array}{lcl}\sigma^{\gamma,\varepsilon}_k &=& \Sigma^{\gamma,\varepsilon\,*}(u_{k-1},y^\varepsilon_k)\,\sigma^{\gamma,\varepsilon}_{k-1}\\[2ex] \sigma^{\gamma,\varepsilon}_0 &=& \rho^\varepsilon,\end{array}\right. \quad (23)$$

where the $*$ denotes adjoint operator. This equation is infinite dimensional, and it is the analog of the Duncan-Mortensen-Zakai discrete time evolution of the conditional density in risk-neutral stochastic control.

We can also define an adjoint process $\nu^{\gamma,\varepsilon}_k$ by the standard duality between measures and functions and the adjoint relationships

$$\begin{array}{lcl}\langle\Sigma^{\gamma,\varepsilon\,*}\sigma,\,\nu\rangle &=& \langle\sigma,\,\Sigma^{\gamma,\varepsilon}\nu\rangle,\\[2ex] \langle\sigma^{\gamma,\varepsilon}_k,\,\nu^{\gamma,\varepsilon}_k\rangle &=& \langle\sigma^{\gamma,\varepsilon}_{k-1},\,\nu^{\gamma,\varepsilon}_{k-1}\rangle\end{array}$$

.

Remark 3.2: The reason for introducing the information state $\sigma^{\gamma,\varepsilon}_k$ is to replace the original output feedback risk-sensitive stochastic control problem with an equivalent stochastic control problem with a state variable $\sigma^{\gamma,\varepsilon}_k$ which is completely observed, and to solve this new problem using dynamic programming. This will yield a state feedback controller for the new problem, or equivalently, an output feedback controller for the original problem which is *separated* through the information state [10], [3], [8], [36], [37], [38].

As in [8], [38], the cost function can be expressed purely in terms of the information state:

$$J^{\gamma,\varepsilon}(u) \;=\; \mathbf{E}^\dagger\left[\langle\sigma^{\gamma,\varepsilon}_M,\,1\rangle\right]. \quad (24)$$

E. Dynamic Programming

Consider the state $\sigma^{\gamma,\varepsilon}$ on the interval $k,\ldots,M$ with initial condition $\sigma^{\gamma,\varepsilon}_k = \sigma$:

$$\left\{\begin{array}{lcl}\sigma^{\gamma,\varepsilon}_l &=& \Sigma^{\gamma,\varepsilon\,*}(u_{l-1},y^\varepsilon_l)\,\sigma^{\gamma,\varepsilon}_{l-1}, \quad k+1\le l\le M,\\[2ex] \sigma^{\gamma,\varepsilon}_k &=& \sigma.\end{array}\right. \quad (25)$$

The corresponding value function for this control problem is defined for σ by

$$S^{\gamma,\varepsilon}(\sigma,k) \;=\; \inf_{u\in\mathcal{O}_{k,M-1}}\mathbf{E}^\dagger\left[\langle\sigma^{\gamma,\varepsilon}_M,\,1\rangle \mid \sigma^{\gamma,\varepsilon}_k = \sigma\right]. \quad (26)$$

The dynamic programming equation for this problem is as follows [8], [38]:

$$\left\{\begin{array}{lcl}S^{\gamma,\varepsilon}(\sigma,k) &=& \inf_{u\in U}\mathbf{E}^\dagger\left[S^{\gamma,\varepsilon}(\Sigma^{\gamma,\varepsilon\,*}(u,y^\varepsilon_{k+1})\sigma,k+1)\right]\\[2ex] S^{\gamma,\varepsilon}(\sigma,M) &=& \langle\sigma,\,1\rangle.\end{array}\right. \quad (27)$$

The next theorem is a statement of the dynamic programming solution to the output feedback risk-sensitive stochastic control problem.

Theorem 3.3: [8], [38] The value function $S^{\gamma,\varepsilon}$ defined by (26) is the unique solution to the dynamic programming equation (27). Conversely, assume that $S^{\gamma,\varepsilon}$ is the solution of the dynamic programming equation (27). Suppose that $u^* \in \mathcal{O}_{0,M-1}$ is a policy such that, for each $k = 0,\ldots,M-1$, $u^*_k = \bar{u}^*_k(\sigma^{\gamma,\varepsilon}_k)$, where $\bar{u}^*_k(\sigma)$ achieves the minimum in (27). Then u^* is an optimal output feedback controller for the risk–sensitive stochastic control problem (§III-C).

Remark 3.4: Note that the controller u^*_k is defined as a function of the information state $\sigma^{\gamma,\varepsilon}_k$, and since $\sigma^{\gamma,\varepsilon}_k$ is a non-anticipating function of $y^\varepsilon_{0,k}$, u^*_k is an output feedback controller for the risk-sensitive stochastic control problem; indeed, u^* is an information state feedback controller.

F. Small Noise Limit

In [8], [36], [37], [38] it was shown that a deterministic dynamic game problem is obtained as a small noise limit of a risk–sensitive stochastic control problem. In this subsection, we carry out this limit procedure for the risk-sensitive stochastic control problem defined above. We first obtain a limit for the information state, and use this to evaluate the appropriate limit for the value function. This yields an information state and value function for the dynamic game problem of Section II-D. These results will be used in Section IV in the solution of the output feedback robust control problem of Section II.

Define the operator kernel $\Lambda^\gamma(u,y'')$ by its entries

$$\Lambda^\gamma(u,y'')_{x,x''} \;\stackrel{\triangle}{=}\; \phi_z(\zeta(x,u)) - \gamma\left(U(x,x'';u) + V(x,y'',u)\right). \quad (28)$$

Then we have

$$\lim_{\varepsilon\to 0}\gamma\varepsilon\log\Sigma^{\gamma,\varepsilon}(u,y'')_{x,x''} \;=\; \Lambda^\gamma(u,y'')_{x,x''}. \quad (29)$$

The action of $\Lambda^\gamma(u, y'')$ and its adjoint is defined in terms of maximization operations as follows:

$$\Lambda^{\gamma *}(u, y'')p(x'') \stackrel{\triangle}{=} \max_{x\in\mathbf{R}^n} \{\Lambda^\gamma(u, y'')_{x,x''} + p(x)\},$$

$$\Lambda^{\gamma}(u, y'')q(x) \stackrel{\triangle}{=} \max_{x''\in\mathbf{R}^n} \{\Lambda^\gamma(u, y'')_{x,x''} + q(x'')\}.$$

The duality pairing $\langle \cdot, \cdot \rangle$ is replaced by the "sup-pairing"

$$(p,\, q) \stackrel{\triangle}{=} \sup_{x\in\mathbf{R}^n} \{p(x) + q(x)\}, \qquad (30)$$

and in fact we have

$$\lim_{\varepsilon\to 0} \gamma\varepsilon \log\langle e^{\frac{1}{\gamma\varepsilon}p},\, e^{\frac{1}{\gamma\varepsilon}q}\rangle = (p,\, q). \qquad (31)$$

The actions corresponding to $\Lambda^\gamma(u, y'')$ are "adjoint" in the sense that

$$(\Lambda^{\gamma *}p,\, q) = (p,\, \Lambda^\gamma q). \qquad (32)$$

The limit result for the information state is the following:

Theorem 3.5: [8], [38] We have

$$\begin{array}{rcl} \lim_{\varepsilon\to 0} \gamma\varepsilon \log \Sigma^{\gamma,\varepsilon *}(u, y) e^{\frac{1}{\gamma\varepsilon}p} &=& \Lambda^{\gamma *}(u, y)p, \\ \lim_{\varepsilon\to 0} \gamma\varepsilon \log \Sigma^{\gamma,\varepsilon}(u, y) e^{\frac{1}{\gamma\varepsilon}q} &=& \Lambda^{\gamma}(u, y)q \end{array} \qquad (33)$$

uniformly on the appropriate function space.

In view of this theorem, we define a limit information state and its adjoint by the recursions

$$\left\{ \begin{array}{rcl} p_k^\gamma &=& \Lambda^{\gamma *}(u_{k-1}, y_k)p_{k-1}^\gamma \\ p_0^\gamma &=& -\beta, \end{array} \right. \qquad (34)$$

and

$$\left\{ \begin{array}{rcl} q_{k-1}^\gamma &=& \Lambda^{\gamma}(u_{k-1}, y_k)q_{k}^\gamma \\ q_M^\gamma &=& 0. \end{array} \right. \qquad (35)$$

Note that

$$(p_k^\gamma,\, q_k^\gamma) = (p_{k-1}^\gamma,\, q_{k-1}^\gamma)$$

for all k.

Turning now to the value function, we have:

Theorem 3.6: [8], [38] The function $W^\gamma(p, k)$ defined by

$$W^\gamma(p, k) \stackrel{\triangle}{=} \lim_{\varepsilon\to 0} \gamma\varepsilon \log S^{\gamma,\varepsilon}(e^{\frac{1}{\gamma\varepsilon}p}, k) \qquad (36)$$

exists (i.e. the sequence converges uniformly), is continuous, and satisfies the recursion

$$\left\{ \begin{array}{rcl} W^\gamma(p, k) &=& \inf_{u\in\mathbf{U}} \sup_{y\in\mathbf{R}^p} \{W^\gamma(\Lambda^{\gamma *}(u, y)p, k+1)\} \\ W^\gamma(p, M) &=& (p,\, 0). \end{array} \right. \qquad (37)$$

IV. SOLUTION TO THE ROBUST CONTROL PROBLEM

A. Equivalent Game Problem

We now replace the deterministic output feedback game problem (Section II) with an equivalent deterministic game problem with p_k^γ, defined in Section III-F, as a completely observed state variable. The solution of this new problem will result in an information state feedback controller, and thus an output feedback controller for the original game problem which is *separated* through the information state.

The next theorem shows that the cost function can be expressed in terms of the information state [8], [36], [37], [38].

Theorem 4.1: We have for all $u \in \mathcal{O}_{0,M-1}$

$$J^\gamma(u) = \sup_{y\in\mathbf{R}^{p(M+1)}} \{(p_M^\gamma,\, 0)\}. \qquad (38)$$

B. Dynamic Programming

Consider now the state p^γ on the interval $k, \ldots, M$ with initial condition $p_k^\gamma = p$:

$$\left\{ \begin{array}{rcl} p_l^\gamma &=& \Lambda^{\gamma *}(u_{l-1}, y_l)p_{l-1}^\gamma, \quad k+1 \le l \le M, \\ p_k^\gamma &=& p. \end{array} \right. \qquad (39)$$

The value function is defined by

$$W^\gamma(p, k) = \inf_{u\in\mathcal{O}_{k,M-1}} \sup_{y\in\mathbf{R}^{p(M-k)}} \{(p_M^\gamma,\, 0) \; : \; p_k^\gamma = p\}. \qquad (40)$$

The solution of the game problem is expressed as follows.

Theorem 4.2: The value function $W^\gamma(p, k)$ defined by (40) is the unique solution to the dynamic programming equation (37). Further, if $W^\gamma(p, k)$ is the solution of (37), and if $u^* \in \mathcal{O}_{0,M-1}$ is a policy such that, for each $k = 0, \ldots, M-1$, $u_k^* = \bar{u}_k^*(p_k^\gamma)$, where $\bar{u}_k^*(p)$ achieves the minimum in (37), then u^* is an optimal policy for the output feedback dynamic game problem (Section II-D).

Proof: Standard dynamic programming arguments.

C. Robust Control

The solution to the state feedback robust control problem was expressed in terms of the solution $\bar{f}_k^\gamma(x)$ of a dynamic programming equation, and a state feedback controller $\tilde{u}_k^*(x)$ was obtained. The framework we have developed in this paper allows us to characterize the solution of the output feedback robust control problem in terms of the solution $W^\gamma(p, k)$ of an infinite dimensional dynamic programming equation, and obtain an output feedback controller $\bar{u}_k^*(p_k^\gamma(\cdot\,; y_{1,k}))$. Note that the information state p_k^γ is also the solution of an infinite dimensional dynamic programming equation (34).

Theorem 4.3: (Necessity) Assume that there exists a controller $u^o \in \mathcal{O}_{0,M-1}$ solving the output feedback robust control problem. Then there exists a solution $W^\gamma(p, k)$ of the dynamic programming equation (37) such that $W^\gamma(-\beta, 0) = 0$. (Sufficiency) Assume that there exits a solution $W^\gamma(p, k)$ of the dynamic programming equation

(37) such that $W^\gamma(-\beta, 0) = 0$, and let $\bar{u}_k^*(p)$ be a control value achieving the minimum in (37). Then $\bar{u}_k^*(p_k^\gamma(\cdot\,; y_{1,k}))$ is an output feedback controller which solves the output feedback robust control problem.

V. Conclusions and Discussion

In this paper we showed that the robust output feedback control problem for nonlinear discrete time systems is solvable using deterministic and stochastic models for the uncertainties. The stochastic model was shown to be a HMM (discrete time continuous state) derived via the maximum entropy principle.

For future work, it would be interesting to investigate connections with model complexity and extensions to more general dynamical systems, including hybrid systems.

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