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Robust Control of Semiconductor Manufacturing Processes

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ABSTRACT. This paper considers the control of semiconductor manufacturing processes. We start by considering the run to run control problem. We pose the problem in a risk-sensitive control framework, and derive the recursion for the information state. We then take large deviation type limits, and go on to show that the limiting information state recursion is related to that arising in solving ultimate boundedness control problems. For completeness we present the solution to the deterministic problem. The paper ends with an example involving application of run to run control to end-pointing.

1. Introduction

In recent years there has been considerable interest in the control of semiconductor manufacturing processes. However, the implementation of modern control techniques has been hindered primarily by: (i) lack of on-line wafer state sensors, (ii) momentum of statistical process control (SPC), and (iii) poor process models, due to a lack of understanding of the physics/chemistry of the processes. The lack of on-line wafer state sensors has resulted in a lot of attention being focused on run to run or batch to batch control (starting with the work of [**HSI92**]). Here, one employs monitor wafers and corrects for process shifts/drifts on a lot to lot basis. Opponents of run to run control argue that doing so tends to increase the variance as well as the higher order moments of the error incurred (see for example Deming's discussion of the funnel experiment [**Dem86**]). Finally, lack of understanding of the fundamentals of the processes (and the resulting plant model mismatch) requires the controller to posses sufficient robustness properties.

Another issue that need to be addressed concerns the noise or disturbance statistics. As the wafer size keeps increasing, the cost of monitor wafers on the overall production process increases substantially. This leads to a reduction in the sample size, and one can no longer appeal to the central limit theorem. It has been found that with smaller sample sizes, the distributions of the pertinent deviations are often non-Gaussian [**CMP95**]. Furthermore, in many cases a strong correlation exists between sequential samples. Lastly, batches of the same raw material supplied

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by different vendors have different distributions within the same tolerance bounds. These result in different statistics of the noise influencing the process depending on the source of the raw materials. Here, adding any new vendor would require a complete recharacterization of the process statistics, and recomputation of the controller. This makes implementation of controllers which depend on the noise statistics extremely cumbersome.

In this paper, we present a mathematical framework for designing controllers that addresses these issues. The resulting controller is deterministic, and it carries out boundedness control (also called l^1-optimal control in the linear systems context [DP87]). We approach the problem by first setting up a risk-sensitive stochastic control problem. We choose this formulation since the cost includes contributions due to the variance, as well as, other higher order moments. Motivated by the fact that although the noise statistics change, the tolerance bounds remain the same, we break up the noise into two components. One taking values in a compact set, and the other being Gaussian with variance ε. The idea being that the noise distributions within the tolerance bounds is represented by the first (in general non-Gaussian) component, and the Gaussian part is employed to account for outliers. In particular, if the tolerance limits have been chosen correctly, then the probability of observing outliers should be extremely small. We set up the information state recursion for this problem. In order to get independence from the noise statistics, we take a large deviations type limit with the variance ε of the Gaussian component going to zero (the case of no outliers). The resulting information state dynamics are independent of the noise statistics, and are a function of the tolerance limits alone. Finally, these dynamics are linked to those arising in the solution of a boundedness control problem.

2. Risk-Sensitive Control

In this section, we consider a special case of the risk-sensitive control problem. On a probability space $(\Omega, \mathcal{F}, \mathbf{P}^u)$ consider the stochastic control problem

$$x^\varepsilon_{k+1} = \zeta^\varepsilon_k + w^\varepsilon_{k+1},\ \zeta^\varepsilon_k \in F(x^\varepsilon_k, u_k)$$
$$y^\varepsilon_{k+1} = \nu^\varepsilon_k + v^\varepsilon_{k+1},\ \nu^\varepsilon_k \in G(x^\varepsilon_k)$$

on the finite time interval $k = 0, \dots, K-1$. The process $y^\varepsilon \in \mathbb{R}$ is measured and is called the observation process. $x^\varepsilon \in \mathbb{R}^n$ represent the states. For convenience, we will write the dynamics as

$$x^\varepsilon_{k+1} \in F(x^\varepsilon_k, u_k) + w^\varepsilon_{k+1}$$
$$y^\varepsilon_{k+1} \in G(x^\varepsilon_k) + v^\varepsilon_{k+1}.$$

Denote by $s_{k,k+j}$, the sequence $\{s_k, s_{k+1}, \dots, s_{k+j}\}$. Let $\mathcal{G}_k$, $\mathcal{Y}_k$ denote the complete filtrations generated by $(x^\varepsilon_{0,k}, y^\varepsilon_{0,k})$ and $y^\varepsilon_{0,k}$ respectively. (We could have more generally considered $G(x,u)$ instead of $G(x)$. Doing so would not affect the results, and the analysis would follow through identically.) We assume that

A1. The controls u_k take values in $U \subset \mathbb{R}^m$, assumed compact and are $\mathcal{Y}_k$ measurable.

With a slight abuse of notation, at time k let $U(k)$ denote the set of control functions u_k which satisfy assumption A1. For $j \geq 0$, we write $U_{k,k+j} = U(k) \bigcup U(k+1) \cdots \bigcup U(k+j)$. For $\mu > 0$, the cost function for the risk-sensitive stochastic control problem is defined for admissible $u \in U_{0,K-1}$ by

$$J^{\mu,\varepsilon}(\rho, u) = \mathbf{E}^u \left[\exp\left(\frac{\mu}{\varepsilon} \sum_{k=1}^{K} L(x^\varepsilon_k, w^\varepsilon_k, u_{k-1}) \right) \right]$$

(ρ defined in A8 below) and the partially observed risk-sensitive stochastic control problem is to find $u^* \in U_{0,K-1}$ such that

$$J^{\mu,\varepsilon}(\rho, u^*) = \inf_{u \in U_{0,K-1}} J^{\mu,\varepsilon}(\rho, u).$$

Let $|\cdot|$ denote the Euclidean norm. Before proceeding further, we state all the assumptions for this section:

A2. $y^\varepsilon_0 = 0$

A3. $\{w^\varepsilon_k\}$ is a $\mathbb{R}^n$-valued i.i.d. noise sequence with density

$$\psi^\varepsilon(w) = (2\pi\varepsilon)^{-n/2} \exp\left(-\frac{1}{2\varepsilon} |w|^2 \right).$$

A4. $\{v^\varepsilon_k\}$ is a real-valued i.i.d. noise sequence with density

$$\phi^\varepsilon(v) = (2\pi\varepsilon)^{-1/2} \exp\left(-\frac{1}{2\varepsilon} |v|^2 \right),$$

independent of $\{w^\varepsilon_k\}$.

A5. $\{\zeta^\varepsilon_k\}$ is a $\mathbb{R}^n$-valued random sequence with $\zeta^\varepsilon_k \in F(x^\varepsilon_k, u_k)$, having a density function $f(x^\varepsilon_k, u_k, \zeta)$ for each k. Furthermore, for each k, ζ^ε_k is independent of w^ε_j and v^ε_j, $j = k+1, \dots, K$. Similarly, $\{\nu^\varepsilon_k\}$ is a $\mathbb{R}$-valued random sequence with $\nu^\varepsilon_k \in G(x^\varepsilon_k)$ having a density function $g(x^\varepsilon_k, \nu)$ for each k. Furthermore, for each k, ν^ε_k is independent of ζ^ε_j, w^ε_{j+1}, v^ε_{j+1} for $j = k, \dots, K-1$. Here, the independence of ν^ε_k, and ζ^ε_k is interpreted as follows. Let x^ε_k and u_k be given. Then ζ^ε_k and ν^ε_k are generated according to their respective probability distributions, independently of each other.

A6. F is a set-valued map from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$, uniformly continuous in x, uniformly in $u \in U$. G is a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}$, satisfying the same assumptions as F. For the definition and properties of uniform continuous set-valued maps, we refer the reader to **[AF90]**.

A7. Furthermore, F, G assume compact values and have a non-empty interior for all x and u. f, and g are bounded functions of their arguments.

A8. x^ε_0 has density $\rho(x) = (2\pi)^{-n/2} \exp(\frac{-1}{2}|x|^2)$.

A9. $L \in C(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m)$ is single-valued, nonnegative, bounded and uniformly continuous.

2.1. Change of Measure. Before proceeding with the solution of the risk-sensitive control problem, we first carry out a change of measure. Using an idea from [EM93], suppose there exists a reference measure $\mathbf{P}^\dagger$, such that under $\mathbf{P}^\dagger$, $\{y^\varepsilon_k\}$ is i.i.d. with density ϕ^ε, independent of $\{x^\varepsilon_k\}$ where x^ε satisfies

$$x^\varepsilon_{k+1} \in F(x^\varepsilon_k, u_k) + w^\varepsilon_{k+1}.$$

Define

$$\Lambda^\varepsilon_k = \Pi^k_{j=1} \left(\int_{\xi \in G(x^\varepsilon_{j-1})} g(x^\varepsilon_{j-1}, \xi) \phi^\varepsilon(v^\varepsilon_j + \xi) d\xi / \phi^\varepsilon(v^\varepsilon_j) \right)$$

and define $\mathbf{P}^\dagger$ by setting

$$\left.\frac{d\mathbf{P}^\dagger}{d\mathbf{P}^u}\right|_{\mathcal{G}_k} = \Lambda_k^\varepsilon$$

i.e. by setting the Radon-Nikodym derivative, restricted to $\mathcal{G}_k$ to equal Λ_k^ε. Note that in general $\mathbf{P}^\dagger$ at k, may depend on the states $x_{0,k-1}^\varepsilon$ (but not on x_k^ε), however we hide this to prevent notational clutter. We write $\mathbf{E}^\dagger$, $\mathbf{E}^u$ to denote expectations with respect to the measures $\mathbf{P}^\dagger$, $\mathbf{P}^u$ respectively. Then

LEMMA 2.1. *Under $\mathbf{P}^\dagger$, the random variables $\{y_j^\varepsilon\}$ are i.i.d. with density function ϕ^ε.*

PROOF. Let $t \in \mathbb{R}$, and consider

$$\begin{aligned}\mathbf{P}^\dagger(y_k^\varepsilon \leq t|\mathcal{G}_{k-1}) &= \mathbf{E}^\dagger\left[I(y_k^\varepsilon \leq t)\,|\mathcal{G}_{k-1}\right] \\ &= \mathbf{E}^u\left[\Lambda_k^\varepsilon I(y_k^\varepsilon \leq t)\,|\mathcal{G}_{k-1}\right]/\mathbf{E}^u\left[\Lambda_k^\varepsilon\,|\mathcal{G}_{k-1}\right].\end{aligned}$$

Now

$$\begin{aligned}\mathbf{E}^u\left[\Lambda_k^\varepsilon\,|\mathcal{G}_{k-1}\right] &= \Lambda_{k-1}^\varepsilon \int_{\mathbb{R}}\int_{G(x_{k-1}^\varepsilon)} g(x_{k-1}^\varepsilon,\xi)\phi^\varepsilon(v_k^\varepsilon+\xi)d\xi dv_k^\varepsilon \\ &= \Lambda_{k-1}^\varepsilon \int_{G(x_{k-1}^\varepsilon)}\int_{\mathbb{R}} g(x_{k-1}^\varepsilon,\xi)\phi^\varepsilon(y_k^\varepsilon)dy_k^\varepsilon d\xi\end{aligned}$$

by changing the order of integration and a change of variables

$$= \Lambda_{k-1}^\varepsilon$$

and

$$\begin{aligned}\mathbf{E}^u\left[\Lambda_k^\varepsilon I(y_k^\varepsilon \leq t)\,|\mathcal{G}_{k-1}\right] &= \Lambda_{k-1}^\varepsilon \int_{\mathbb{R}}\int_{G(x_{k-1}^\varepsilon)} g(x_{k-1}^\varepsilon,\xi)I(y_k^\varepsilon \leq t)\phi^\varepsilon(v_k^\varepsilon+\xi)d\xi dv_k^\varepsilon \\ &= \Lambda_{k-1}^\varepsilon \int_{G(x_{k-1}^\varepsilon)}\int_{\mathbb{R}} I(y_k^\varepsilon \leq t) g(x_{k-1}^\varepsilon,\xi)\phi^\varepsilon(y_k^\varepsilon)dy_k^\varepsilon d\xi\end{aligned}$$

by changing the order of integration, and a change of variables.

$$= \Lambda_{k-1}^\varepsilon \int_{-\infty}^{t} \phi^\varepsilon(y_k^\varepsilon)dy_k^\varepsilon.$$

The result follows. □

It is clear that under $\mathbf{P}^\dagger$, y_j^ε, and x_j^ε are independent. Furthermore, the existence of $\mathbf{P}^\dagger$ is guaranteed by Kolmogorov's extension theorem. In a similar manner, we define the inverse transformation relating $\mathbf{P}^u$ to $\mathbf{P}^\dagger$ as follows.

$$\left.\frac{d\mathbf{P}^u}{d\mathbf{P}^\dagger}\right|_{\mathcal{G}_k} = \mathcal{Z}_k^\varepsilon = \Pi_{j=1}^k \Psi^\varepsilon(x_{j-1}^\varepsilon, y_j^\varepsilon)$$

where

$$\begin{aligned}\Psi^\varepsilon(x_{j-1}^\varepsilon, y_j^\varepsilon) &\triangleq \int_{G(x_{j-1}^\varepsilon)} g(x_{j-1}^\varepsilon,\xi)\phi^\varepsilon(y_j^\varepsilon-\xi)d\xi/\phi^\varepsilon(y_j^\varepsilon) \\ &= \int_{G(x_{j-1}^\varepsilon)} g(x_{j-1}^\varepsilon,\xi)\exp\left(-\frac{1}{\varepsilon}(\frac{1}{2}|\xi|^2 - y_j^\varepsilon\xi)\right)d\xi.\end{aligned}$$

2.2. Information State. Consider the space $L^\infty(\mathbb{R}^n)$ and its dual $L^{\infty*}(\mathbb{R}^n)$. We will denote the natural bilinear pairing between $L^\infty(\mathbb{R}^n)$ and $L^{\infty*}(\mathbb{R}^n)$ by $<\tau,\eta>$ for $\tau \in L^{\infty*}(\mathbb{R}^n)$, $\eta \in L^\infty(\mathbb{R}^n)$.

We define the information state process $\sigma_k^{\mu,\varepsilon} \in L^{\infty*}(\mathbb{R}^n)$ by

$$<\sigma_k^{\mu,\varepsilon},\eta> = \mathbf{E}^\dagger[\eta(x_k^\varepsilon)\exp(\frac{\mu}{\varepsilon}\sum_{j=1}^k L(x_j^\varepsilon, w_j^\varepsilon, u_{j-1}))\mathcal{Z}_k^\varepsilon|\mathcal{Y}_k]$$

for all test functions $\eta \in L^\infty(\mathbb{R}^n)$, for $k = 1,\ldots,K$, with $\sigma_0^{\mu,\varepsilon} = \rho \in L^1(\mathbb{R}^n)$. We introduce the bounded linear operator $\Xi^{\mu,\varepsilon} : L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ defined by

$$\begin{aligned}\Xi^{\mu,\varepsilon}(u,y)\eta(\xi) \triangleq \int_{\mathbb{R}^n}\int_{F(\xi,u)} &\psi^\varepsilon(z-r)\exp\left(\frac{\mu}{\varepsilon}L(z,z-r,u)\right)f(\xi,u,r)\eta(z)\cdot \\ &\Psi^\varepsilon(\xi,y)drdz\end{aligned}$$

and its adjoint $\Xi^{\mu,\varepsilon*} : L^{\infty*}(\mathbb{R}^n) \to L^{\infty*}(\mathbb{R}^n)$ defined by

$$\begin{aligned}\Xi^{\mu,\varepsilon*}(u,y)\sigma(z) \triangleq \int_{\mathbb{R}^n}\int_{F(\xi,u)} &f(\xi,u,r)\psi^\varepsilon(z-r)\exp\left(\frac{\mu}{\varepsilon}L(z,z-r,u)\right)\cdot \\ &\Psi^\varepsilon(\xi,y)\sigma(\xi)drd\xi.\end{aligned} \tag{2.1}$$

LEMMA 2.2. *The information state $\sigma_k^{\mu,\varepsilon}$ satisfies*

$$\begin{cases}\sigma_k^{\mu,\varepsilon} = \Xi^{\mu,\varepsilon*}(u_{k-1},y_k^\varepsilon)\sigma_{k-1}^{\mu,\varepsilon} \\ \sigma_0^{\mu,\varepsilon} = \rho,\ k = 1,\ldots,K.\end{cases} \tag{2.2}$$

PROOF.

$$\begin{aligned}<\sigma_k^{\mu,\varepsilon},\eta> &= \mathbf{E}^\dagger[\eta(x_k^\varepsilon)\exp(\frac{\mu}{\varepsilon}\sum_{j=1}^k L(x_j^\varepsilon,w_j^\varepsilon,u_{j-1}))\mathcal{Z}_k^\varepsilon|\mathcal{Y}_k] \\ &= \mathbf{E}^\dagger[\eta(x_k^\varepsilon)\exp(\frac{\mu}{\varepsilon}L(x_k^\varepsilon,w_k^\varepsilon,u_{k-1}))\Psi^\varepsilon(x_{k-1}^\varepsilon,y_k^\varepsilon)\cdot \\ &\qquad \exp(\frac{\mu}{\varepsilon}\sum_{j=1}^{k-1}L(x_j^\varepsilon,w_j^\varepsilon,u_{j-1}))\mathcal{Z}_{k-1}^\varepsilon|\mathcal{Y}_k] \\ &= \mathbf{E}^\dagger[\int_{\mathbb{R}^n}\int_{F(x_{k-1}^\varepsilon,u_{k-1})}\eta(z)\exp(\frac{\mu}{\varepsilon}L(z,z-r,u_{k-1}))\Psi^\varepsilon(x_{k-1}^\varepsilon,y_k^\varepsilon)\cdot \\ &\qquad f(x_{k-1}^\varepsilon,u_{k-1},r)\exp(\frac{\mu}{\varepsilon}\sum_{j=1}^{k-1}L(x_j^\varepsilon,w_j^\varepsilon,u_{j-1}))\mathcal{Z}_{k-1}^\varepsilon\cdot \\ &\qquad \psi^\varepsilon(z-r)drdz|\mathcal{Y}_k] \\ &=<\sigma_{k-1}^\varepsilon,\int_{\mathbb{R}^n}\int_{F(\cdot,u_{k-1})}\eta(z)\exp(\frac{\mu}{\varepsilon}L(z,z-r,u_{k-1}))f(\cdot,u_{k-1},r)\cdot \\ &\qquad \Psi^\varepsilon(\cdot,y_k^\varepsilon)\psi^\varepsilon(z-r)drdz> \\ &=<\sigma_{k-1}^\varepsilon,\Xi^{\mu,\varepsilon}(u_{k-1},y_k^\varepsilon)\eta> \\ &=<\Xi^{\mu,\varepsilon*}(u_{k-1},y_k^\varepsilon)\sigma_{k-1}^\varepsilon,\eta>\end{aligned}$$

for any $\eta \in L^\infty(\mathbb{R}^n)$. □

Observe that for all $u \in U_{0,K-1}$, we have

$$\begin{aligned}\mathbf{E}^\dagger[<\sigma_K^{\mu,\varepsilon},1>] &= \mathbf{E}^\dagger[\mathbf{E}^\dagger[\exp(\frac{\mu}{\varepsilon}\sum_{j=1}^{K} L(x_j^\varepsilon, w_j^\varepsilon, u_{j-1}))\mathcal{Z}_K^\varepsilon|\mathcal{Y}_K]] \\ &= \mathbf{E}^\dagger[\exp(\frac{\mu}{\varepsilon}\sum_{j=1}^{K} L(x_j^\varepsilon, w_j^\varepsilon, u_{j-1}))\mathcal{Z}_K^\varepsilon] \\ &= J^{\mu,\varepsilon}(\rho, u).\end{aligned}$$

Thus, the cost can be expressed as a function of $\sigma_K^{\mu,\varepsilon}$ alone, and hence the name *information state* for $\sigma_k^{\mu,\varepsilon}$ is justified. We can now obtain the solution to the risk-sensitive stochastic control problem via dynamic programming. This methodology is well known in the stochastic control literature [**JBE94**],[**EM93**]. Define the value function for $\sigma \in L^1(\mathbb{R}^n)$ by

$$V^{\mu,\varepsilon}(\sigma,k) = \inf_{u\in U_{k,K-1}} \mathbf{E}^\dagger[<\sigma_K^{\mu,\varepsilon},1> | \sigma_k^{\mu,\varepsilon}=\sigma].$$

Then it can be shown that this satisfies the following dynamic programming equation

$$\begin{aligned}V^{\mu,\varepsilon}(\sigma,k) &= \inf_{u\in U_{k,k}} \mathbf{E}^\dagger[V^{\mu,\varepsilon}(\Xi^{\mu,\varepsilon*}(u,y_{k+1})\sigma, k+1)] \\ V^{\mu,\varepsilon}(\sigma,K) &= \mathbf{E}^\dagger[<\sigma,1>]\end{aligned}$$

for $k = 0,\ldots,K-1$, where the infimizing control value $u_k(\sigma)$ solves the risk-sensitive control problem. It is clear that u_k (the control value at time k) is a function only of (the information state) $\sigma_k^{\mu,\varepsilon}$ at time k. Hence, the policy is separated, and the information state contains all the relevant information required for control.

The result above shows that the controller which solves the risk-sensitive stochastic control problem is a function of f, and g. As mentioned in the Introduction, these density functions could change frequently. Hence, because the problem needs to be solved repeatedly with the appearance of new noise distributions, a direct application of the results is extremely cumbersome and not practical. However, what do remain constant are the tolerance limits specified for that particular mix of product and raw materials. One can view the set-valued maps F, and G as representatives of these tolerance limits. Furthermore, if the limits have been properly specified, then the probability of outliers is extremely small. We employ this fact to motivate the next subsection, where we take "small noise" limits of the Gaussian component of the noise. As will be observed, under these large deviations type limits the resulting information state dynamics are in fact independent of the actual noise distributions. Furthermore, we will see in the next section the results can be related to the solution of a l_1-optimal control type problem (however in our case for general nonlinear systems). A further advantage of the deterministic interpretation is that we can drop most of the restrictive assumptions associated with the risk-sensitive control problem, in particular the strong independence assumptions.

2.3. Small Noise Limits. We first define some spaces following [**JBE94**]. For $\gamma \in M \triangleq \{\gamma \in \mathbb{R}^2 \mid \gamma_1 > 0, \gamma_2 \geq 0\}$ define

$$\mathcal{D}^\gamma \triangleq \{\tilde{p} \in C(\mathbb{R}^n) \mid \tilde{p}(x) \leq -\gamma_1 \mid x \mid^2 + \gamma_2\}$$

$$\mathcal{D} \triangleq \{\tilde{p} \in C(\mathbb{R}^n) \mid \tilde{p}(x) \leq -\gamma_1 \mid x \mid^2 + \gamma_2 \text{ for some } \gamma \in M\}.$$

We equip these spaces with the topology of uniform convergence on compact subsets. Define $\Lambda^{\mu*} : \mathcal{D} \to \mathcal{D}$ by

$$\begin{aligned}\Lambda^{\mu*}(u,y)\tilde{p}(z) \triangleq \sup_{\xi\in\mathbb{R}^n}\{\tilde{p}(\xi) + \sup_{r\in F(\xi,u)} (L(z, z-r, u) - \frac{1}{2\mu} \mid z - r \mid^2) - \\ \frac{1}{\mu}\inf_{s\in G(\xi)}(\frac{1}{2} \mid s \mid^2 - sy)\}\end{aligned}$$

for $\tilde{p} \in \mathcal{D}$.

Then we have

THEOREM 2.3.

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\mu} \log \Xi^{\mu,\varepsilon*}(u,y)e^{\frac{\mu}{\varepsilon}\tilde{p}} = \Lambda^{\mu*}(u,y)\tilde{p}$$

in $\mathcal{D}$ uniformly on compact subsets of $U \times \mathbb{R} \times \mathcal{D}^\gamma$ for each $\gamma \in M$.

PROOF. From (2.1) we have

$$\begin{aligned}\frac{\varepsilon}{\mu}\log \Xi^{\mu,\varepsilon*}(u,y)e^{\frac{\mu}{\varepsilon}\tilde{p}}(z) = \frac{\varepsilon}{\mu}\log \int_{\mathbb{R}^n}\int_{F(\xi,u)}\int_{G(\xi)} \exp\frac{\mu}{\varepsilon}(\frac{-1}{2\mu} \mid z-r \mid^2 - \\ \frac{n\varepsilon}{2\mu}\log(2\pi\varepsilon) + L(z, z-r, u) + \tilde{p}(\xi) + \frac{\varepsilon}{\mu}\log f(\xi,u,r) + \\ \frac{\varepsilon}{\mu}\log g(\xi,s) - \frac{1}{\mu}[\frac{1}{2} \mid s \mid^2 - sy])dsdrd\xi.\end{aligned}$$

Under the assumptions made on the system, a straightforward application of the Varadhan-Laplace lemma (Appendix) yields the result. □

REMARK 2.4. In particular, setting $\sigma^{\mu,\varepsilon} = e^{\frac{\mu}{\varepsilon}\tilde{p}}$ in equation (2.2), and employing the result of theorem 2.3, we obtain

$$\begin{aligned}\tilde{p}_{k+1}(z) = \sup_{\xi\in\mathbb{R}^n}\{\tilde{p}_k(\xi) + \sup_{r\in F(\xi,u_k)} (L(z, z-r, u_k) - \frac{1}{2\mu} \mid z-r \mid^2) - \\ \frac{1}{\mu}\inf_{s\in G(\xi)}(\frac{1}{2} \mid s \mid^2 - sy_{k+1})\}\end{aligned} \tag{2.3}$$

for $k = 0,\ldots,K-1$.

REMARK 2.5. If we had G as a function of both $\xi \in \mathbb{R}^n$, and $u \in U$, all the results would follow through, provided G, and g satisfy the same assumptions as F and f respectively. For the deterministic case (next section), we will employ $G(x,u)$ instead of $G(x)$.

3. Deterministic Problem

We now consider the deterministic system (corresponding to the no outliers case) defined by

$$\Sigma \begin{cases} x_{k+1} \in F(x_k, u_k), & x_0 \in X_0 \\ y_{k+1} \in G(x_k, u_k) \end{cases} \tag{3.1}$$

for $k = 0,\ldots,K-1$. Here, X_0 denotes the set of possible initial conditions. We assume that the system (3.1) satisfies the relevant assumptions of section 2. Namely, that F, G take on compact values with non-empty interior, and $u_k \in U$, with U compact. We first simplify the information state recursion (2.3) for this case. Here it is assumed that we have access to the function L, which is tied to the particular kind of robust control problem being considered. More will be said about this in the next subsection.

We carry out the following change of variables in equation (2.3)

$$p_0(x) \triangleq \tilde{p}_0(x)$$
$$p_k(x) \triangleq \tilde{p}_k(x) - \frac{1}{2\mu}\sum_{j=0}^{k-1} | y_{j+1} |^2 \ , \ k = 1, \dots, K.$$

Then equation (2.3) can be written as

$$p_{k+1}(x) = \sup_{\xi\in\Re^n} \{p_k(\xi) + \sup_{r\in F(\xi,u_k)} (L(x, x-r, u_k) - \frac{1}{2\mu}|x-r|^2) - \frac{1}{2\mu}\inf_{s\in G(\xi,u_k)} (s - y_{k+1})^2.\} \tag{3.2}$$

Using the convention that the supremum over an empty set is $-\infty$, we can place a natural restriction on ξ. Define

$$\Omega(x, y, u) \triangleq \{\xi \in \mathbb{R}^n \mid x \in F(\xi, u) \text{ and } y \in G(\xi, u)\}.$$

This just ensures that the values of ξ are compatible with x, u and y, given the dynamics (3.1) (i.e. there are no outliers). Then equation (3.2) can be written as

$$p_{k+1}(x) = \sup_{\xi\in\Omega(x,y_{k+1},u_k)} \{p_k(\xi) + \sup_{r\in F(\xi,u_k)} (L(x, x-r, u_k) - \frac{1}{2\mu} | x - r |^2)\} \tag{3.3}$$

or by (compactly) writing $H(p_k, y_{k+1}, u_k)(x)$ for the right hand side of (3.3) as

$$p_{k+1} = H(p_k, y_{k+1}, u_k)$$
$$p_0 = \bar{p}$$

yielding the information state recursion for the deterministic system. Here, $\bar{p}$ denotes a weighting on the initial states. Since, we know that $x_0 \in X_0$, we can set $\bar{p}(x) = -\infty$ for all $x \notin X_0$.

We next define

$$\Gamma^u_{0,k}(x_0) \triangleq \{x_{0,k} \in \mathbb{R}^n_{0,k} | x_{j+1} \in F(x_j, u_j), j = 0, \dots, k-1\}$$

where $\mathbb{R}^n_{0,k} = \{x_{0,k} | x_j \in \mathbb{R}^n, j = 0, \dots, k\}$, and

$$\Gamma^{u,y}_{0,k}(x_0) \triangleq \{x_{0,k} \in \Gamma^u_{0,k}(x_0) | y_{j+1} \in G(x_j, u_j), j = 0, \dots, k-1\}.$$

Furthermore, we write $r, s \in \Gamma^u_{0,k}(x_0)$ for trajectories r and s such that $r \in \Gamma^u_{0,k}(x_0)$, and $s_{j+1} \in F(r_j, u_j)$, for $j = 0, \dots, k-1$, with $s_0 = r_0 = x_0$. We similarly write $r, s \in \Gamma^{u,y}_{0,k}(x_0)$. Consider the information state recursion (3.3). By inspection, one obtains

$$p_k(x) = \sup_{x_0\in X_0} \sup_{r,s\in\Gamma^{u,y}_{0,k}(x_0)} \{\bar{p}(x_0) + \sum_{j=0}^{k-1} L(r_{j+1}, r_{j+1} - s_{j+1}, u_j) - \frac{1}{2\mu}|r_{j+1} - s_{j+1}|^2 | \ r_k = x\}. \tag{3.4}$$

Here, p_k can be interpreted as the *worst case cost to come* for a dynamic game problem. The game in this case is between the controller, and the set of state

trajectories that the system can generate (which in turn is controlled by the disturbances). We illustrate in the next subsection that this game is related to the ultimate boundedness control (l^1-optimal control for linear systems) problem.

3.1. Boundedness Control. Consider the system Σ (3.1), along with a regulated output

$$z_{k+1} = h(x_{k+1}, u_k)$$

on an infinite horizon $k = 0, 1, 2, \dots$, with $z_{k+1} \in \mathbb{R}^l$. Note that in general the (apparently strange) indexing of u poses no problems, as we can always pass the regulated output through a strictly proper low pass filter. Assume that h is such that for any $\gamma > 0$, the set $\mathcal{L}^\gamma$ defined by

$$\mathcal{L}^\gamma \triangleq \{x \in \mathbb{R}^n | \exists u \in U \text{ s.t. } |h(x, u)| \le \gamma\}$$

is compact and contains the origin. Furthermore, assume that 0 is an equilibrium point of Σ, i.e.

$$F(0,0) \ni 0, \qquad G(0,0) \ni 0, \qquad h(0,0) = 0.$$

Let O be the space of output feedback policies, i.e. policies such that $u_k = \tilde{u}(u_{0,k-1}, y_{1,k})$. The ultimate boundedness control problem can be stated as: Given $\gamma = \frac{1}{\sqrt{2\mu}} > 0$, find an output feedback policy $u \in O$, such that

C1. If $x_0 = 0$, then $|z_k| \le \gamma$ for all $k = 0, 1, 2, \dots$

C2. If $x_0 \ne 0$, then

$$\limsup_{k\to\infty} |z_k| \le \gamma.$$

Our assumptions on h ensure that is C1 and C2 are satisfied, then all state trajectories are ultimately bounded. Hence, we need only worry about satisfaction of C1 and C2 (i.e. the performance criteria). In particular, z_k could be the tracking or regulation error and the objective here is to ensure that if the system starts from rest (zero initial conditions) it remains bounded by γ in the presence of persistent disturbances or else it tends asymptotically to this bound. We define a new function $l : \mathbb{R}^n \times U \to \mathbb{R}$

$$\bar{z}_{k+1} = l(x_{k+1}, u_k) = \int_0^{x_{k+1}} |h(\xi, u_k)| d\xi.$$

Before proceeding further, define the "sup-pairing" (p, q) as

$$(p, q) = \sup_{x\in\mathbb{R}^n} \{p(x) + q(x)\}.$$

We can now identify L in equation (3.4) to l defined above as

$$L(x, w, u) \triangleq |l(x, u) - l(x - w, u)|^2$$

to obtain

$$p_k(x) = \sup_{x_0\in X_0} \sup_{r,s\in\Gamma^u(x_0)} \{\bar{p}(x_0) + \sum_{j=0}^{k-1} |l(r_{j+1}, u_j) - l(s_{j+1}, u_j)|^2 - \gamma^2 |r_{j+1} - s_{j+1}|^2 | r_k = x\}. \tag{3.5}$$

We now state the solvability of the boundedness control problem in terms of a new cost function involving the information state alone.

LEMMA 3.1. *For any $u \in O$, the closed loop system Σ^u satisfies C1 and C2 if the information state p_k (equation (3.5)) satisfies*

$$\sup_{k\geq 1}\ \sup_{y_{1,k}\in\Delta^u_{1,k}(X_0)} \{(p_k,0)|p_0=-\beta^u\}\leq 0 \tag{3.6}$$

for some $\beta^u(x)\geq 0$, $\beta^u(0)=0$. Here, $\Delta^u_{1,k}(X_0)$ is the set of all measurement trajectories that can be generated by the closed-loop system Σ^u up to time k.

PROOF. For any $k\geq 1$

$$\begin{aligned}\sup_{y_{1,k}\in\Delta^u_{1,k}(X_0)}\{(p_k,0)|p_0=-\beta^u\} &= \sup_{y_{1,k}\in\Delta^u_{1,k}(X_0)}\{\sup_{x\in\mathbb{R}^n}\ \sup_{x_0\in X_0}\ \sup_{r,s\in\Gamma^{u,y}_{0,k}(x_0)}\{-\beta^u(x_0)+\\ &\qquad \sum_{j=0}^{k-1}|l(r_{j+1},u_j)-l(s_{j+1},u_j)|^2-\\ &\qquad\qquad \gamma^2|r_{j+1}-s_{j+1}|^2|r_k=x\}\}\\ &= \sup_{x_0\in X_0}\ \sup_{r,s\in\Gamma^u_{0,k}(x_0)}\{-\beta^u(x_0)+\\ &\qquad \sum_{j=0}^{k-1}|l(r_{j+1},u_j)-l(s_{j+1},u_j)|^2-\\ &\qquad\qquad \gamma^2|r_{j+1}-s_{j+1}|^2\}.\end{aligned}$$

For any k, $x_0\in X_0$, and $r_{0,k},s_{0,k}\in\Gamma^u_{0,k}(x_0)$, we have

$$\sum_{j=0}^{k-1}|l(r_{j+1},u_j)-l(s_{j+1},u_j)|^2-\gamma^2|r_{j+1}-s_{j+1}|^2\leq\beta^u(x_0).$$

Pick

$$\hat{s}_{j+1}\in\arg\max_{s\in F(r_j,u_j)}\{|l(r_{j+1},u_j)-l(s,u_j)|^2-\gamma^2|r_{j+1}-s|^2\},\ j=0,\ldots,k-1,$$

and set

$$Q_j=|l(r_{j+1},u_j)-l(\hat{s}_{j+1},u_j)|^2-\gamma^2|r_{j+1}-\hat{s}_{j+1}|^2.$$

Then $Q_j\geq 0, j=0,1,2,\ldots,k-1$, and

$$\sum_{j=0}^{k-1}Q_j\leq\beta^u(x_0),\ \text{for all } k.$$

Hence,

$$Q_k\to 0 \text{ as } k\to\infty$$

which implies that

$$\limsup_{k\to\infty}|z_k|\leq\gamma.$$

Furthermore, if $x_0=0$, then $Q_k=0, k=0,1,\ldots$, and it follows that

$$|z_k|\leq\gamma,\quad k=0,1,2,\ldots.$$

□

Lemma 3.1 also implies that the policy is separated. In particular, we can write down the solvability of the boundedness control problem solely in terms of the information state system (3.5), where the information states p play the role of the states, and y are the disturbances. Hence, we have converted a partially observed problem to a fully observed one, and the objective now is to find a policy that minimizes the left hand side of (3.6). Such a policy will be a function of the information state alone i.e. $u_k=u(p_k)$, and we call such policies *information state feedback policies*, denoted by $I\subset O$.

We now apply dynamic programming to obtain the control policy. For any $k\geq 1$, define the value function by

$$M_k(p)=\inf_{u\in O_{0,k-1}}\ \sup_{y\in\Delta^u_{1,k}(X_0)}\{(p_k,0)|p_0=p\}$$

then the corresponding dynamic programming equation is

$$M_k(p)=\inf_{u\in U}\sup_{y\in\mathbb{R}}\{M_{k-1}(H(p,u,y))\},\ k=1,2,\ldots$$

with $M_0(p)=(p,0)$. Passing to the limit as $k\to\infty$, and invoking stationarity we obtain the following stationary dynamic programming equation

$$M(p)=\inf_{u\in U}\sup_{y\in\Re}\{M(H(p,u,y))\} \tag{3.7}$$

where $M(p)=\lim_{k\to\infty}M_k(p)$. Let $\mathcal{E}$ denote the space in which p lives. We define for any function $Z:\mathcal{E}\to\mathbb{R}^*$

$$dom\ Z\triangleq\{p\in\mathcal{E}|\ Z(p)\text{ is finite}\}.$$

The following theorem states the solvability condition for the ultimate boundedness control problem in terms of the solvability of (3.7).

THEOREM 3.2. *Assume there exists a solution M to the dynamic programming equation (3.7) on some nonempty domain dom M, such that $-\beta\in dom\ M$ ($\beta(x)\geq 0$, for all $x\in X_0$, $\beta(0)=0$), $M(-\beta)=0$, $M(p)\geq(p,0)$. Let $\bar{u}\in I$ be a policy such that $\bar{u}(p)$ achieves the minimum in (3.7). Let $p_0=-\beta$, and let p_k be the corresponding information state trajectory satisfying $p_k\in dom\ M$, $k=0,1,\ldots$. Then $\bar{u}\in I$ solves the ultimate boundedness control problem.*

PROOF. Since $p_k\in dom\ M$ for all k, we have for any k

$$M(p_k)=\sup_{y\in\Re}M(H(p_k,\bar{u}(p_k),y)).$$

Hence for any observed trajectory $y_{1,k}$, we have

$$(p_k,0)\leq M(p_k)\leq M(p_0)=M(-\beta)=0$$

and hence by lemma 3.1, all trajectories are ultimately bounded. □

REMARK 3.3. During the development of the solution to the boundedness control problem, we made the assumption that there were no outliers. This is what motivated us to define the information state along the lines of (3.3). Allowing for outliers forces us to consider the information state defined by (3.2), but in doing so the risk-sensitive control problem can no longer be linked to the boundedness control problem. This is to be expected, since the boundedness control problem no longer makes sense for the case where we allow outliers (since the system can always violate the bounds on z_k due to outliers).

REMARK 3.4. We could have let $|\cdot|$ represent any norm of our choice without affecting the results for the deterministic problem. Of particular interest here is the ∞ norm (related to l^1-optimal control).

REMARK 3.5. The approach taken here yields a dynamic game based approach to boundedness control (as well as l^1-optimal control) as compared to viability theory [**Aub91**] based approaches of the past [**Sha94**].

The above yields a very intuitive framework for setting up and posing non-linear control design problems e.g. one can easily handle the case of parametric uncertainty. Clearly, solving the stationary dynamic programming equation (3.7) is computationally hard. Hence, in practice we implement a certainty equivalence controller. Note that the cost to be minimized is given by

$$\tilde{J}(u) = \sup_{r,s\in\Gamma^u(x_0)} \{\sum_{k=0}^{\infty} |l(r_{k+1}, u_k) - l(s_{k+1}, u_k)|^2 - \gamma^2 |r_{k+1} - s_{k+1}|^2\}$$

The corresponding stationary dynamic programming equation for the state feedback case is

$$W(x) = \inf_{u\in U} \sup_{r,s\in F(x,u)} \{|l(r,u) - l(s,u)|^2 - \gamma^2 |r-s|^2 + W(r)\} \tag{3.8}$$

with $W(x) \geq 0$, $W(0) = 0$. This yields a state feedback policy u_F such that $u_F(x)$ achieves the infimum in (3.8). The certainty equivalence controller is then implemented as follows. Suppose we are at time k, i.e. we know p_k, we first estimate

$$\hat{x}_k \in \arg\max_{x\in\mathbb{R}^n} \{p_k(x) + W(x)\}$$

and then implement $u_k = u_F(\hat{x}_k)$. The conditions for this controller to optimally solve the dynamic game problem are similar to those arising in the nonlinear H_∞ context [**Jam94**], [**BP96a**].

4. Example

Since the approach taken in the developments above was motivated by run to run control, we focus on this problem for our example. The specific problem is to end-point a deposition process in order to obtain a desired deposition thickness. The problem chosen yields a very simple system, but is very important from the industry point of view. Additional examples of the theory developed include robust control applied to rapid thermal processing [**BP96b**].

4.1. End-Pointing. Lots consisting of 24 wafers are processed through a single wafer reactor. Here, we assume that the process under consideration is deposition. Measurements are carried out on the last wafer of each lot. The aim is to determine the processing time, so as to achieve a given target thickness. Here, it is assumed that the processing time per wafer is constant for all wafers across a lot. We assume that the process is subject to three kinds of noise: (i) variation in the average deposition rate at the test wafer from lot to lot, (ii) variation in the instantaneous rate from test wafer to test wafer, due to changes in both the wafer surface, and deposition conditions, and (iii) measurement noise, either due to finite

resolution of the measurement apparatus, or due to experimental error. Here, the basic process can be modeled as

$$\begin{aligned}\hat{r}_{k+1} &= \hat{r}_k + v_k\\ \hat{\xi}_{k+1} &= (\hat{r}_k + w_k)\hat{t}_k\\ \hat{y}_{k+1} &= (\hat{r}_k + w_k)\hat{t}_k + m_k\end{aligned}$$

where $\hat{r}_k$ is the average deposition rate for lot k, $\hat{\xi}_{k+1}$ is the actual deposition thickness on the test wafer for lot k for a deposition time $\hat{t}_k$, and $\hat{y}_{k+1}$ is the measured thickness. Here, v_k is the noise used to model the variation in the average deposition rate, w_k is the noise used to model the rate variation per wafer, and m_k is the noise modeling the measurement error. The objective is to maintain a deposition thickness of 1500$\AA$. It is also assumed that is $\hat{r}_k$ leaves the interval [375, 800], a maintainence call will be placed (i.e. the deposition rate is too slow, or extremely fast). This is typically done due to the following reasons. A very slow deposition rate is undesirable, since this slows down production. On the other hand a very fast deposition rate adversely affects the properties of the material being deposited (e.g. grain size) and impacts the performance of the final integrated circuit. We let v_k be zero mean, Gaussian with standard deviation 2. w_k is modeled by taking the sum of two random variables, one from a uniform distribution over [−12, 12], and the other being zero mean Gaussian with variance 1. The measurement error is assumed to be uniformly distributed between [−10, 10]. Based on this, we now place bounds on these noise values. Specifically we assume that

$$v_k \in [-6, 6]; \quad w_k \in [-15, 15]; \quad m_k \in [-10, 10].$$

The cost is defined as

$$z_{k+1} = l(x_{k+1}, u_k) = 0.1(\hat{\xi}_{k+1} - 1500)^2.$$

We solve the state feedback problem, assuming that $W(\hat{r}, \hat{\xi}) = 0$ for all $\hat{r} \notin [375, 800]$. We iteratively test different values of γ, and the smallest one for which the state feedback problem is solvable is approximately 20.2. We now implement the certainty equivalence controller. Figure 1 shows the controlled and uncontrolled trajectories (actual deposition $\hat{\xi}$) for the system under drift generated by a Gaussian distribution with mean 3 and variance 1 for v_k. It is observed that the noise induced by the controller is extremely small, however the controller still tracks the target accurately. Finally, Figure 2 shows the controller response to a sudden shift in the deposition rate. The controller corrects for the shift in the very next run. The perturbations in the deposition thickness are magnified for run 21 onwards, since the deposition time is greater for the controlled case and this tends to magnify the effect of w_k.

Appendix

Here, we give an extension of the Varadhan-Laplace lemma presented in [**JBE94**]. Below ρ denotes a metric on $C(\mathbb{R}^n \times \mathbb{R}^p)$ corresponding to uniform convergence on compact subsets. $\mathcal{B}_r(x)$ denotes the open ball centered at x of radius r. $L^a_G(\cdot) : \mathbb{R}^n \to \mathbb{R}^p$ defined as Furthermore, it is assumed that $|\cdot|$ denotes the Euclidean norm. In what follows, F^ε_a, F_a denote single-valued maps, and G^a is a

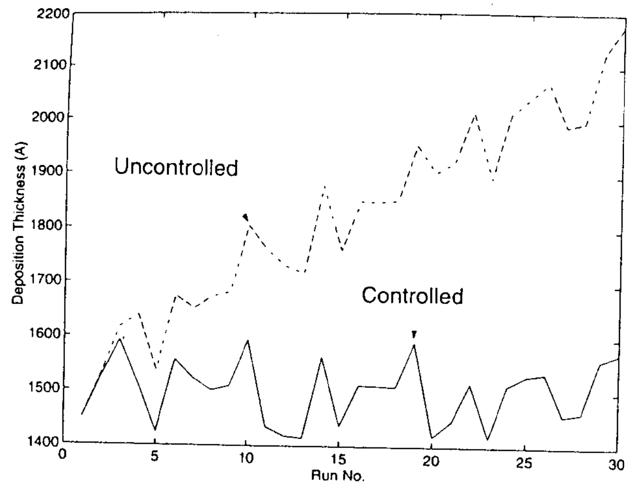

FIGURE 1. End-pointing: Controlled and uncontrolled trajectories for process under drift.

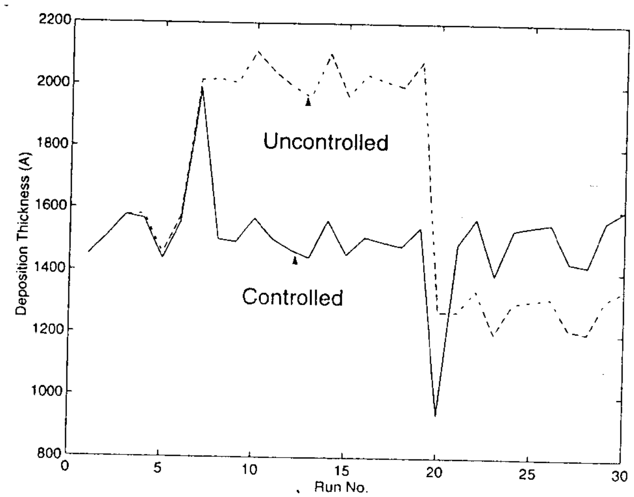

FIGURE 2. End-pointing: Controlled and uncontrolled trajectories for process shifts.

set-valued map. We also define, $L_G^a(M)$ as

$$L_G^a(M) \triangleq \bigcup_{x \in M} G^a(x).$$

LEMMA A.1 (Varadhan-Laplace). *Let A be a compact space, F_a^ε, $F_a \in C(\mathbb{R}^n \times \mathbb{R}^p)$ and assume*

i. $\lim_{\varepsilon \to 0+} \sup_{a \in A} \rho(F_a^\varepsilon, F_a) = 0$
ii. *The function F_a is uniformly continuous in each argument on each set $\mathcal{B}_R(0) \times \mathcal{B}_{\hat{R}}(0)$; $R, \hat{R} > 0$, uniformly in $a \in A$.*
iii. $\exists \gamma_1 > 0$, $\gamma_2 \geq 0$ *such that*
$$F_a^\varepsilon(x,w), F_a(x,w) \leq -\gamma_1 \left(\mid x \mid^2 + \mid w \mid^2\right) + \gamma_2$$
$\forall x \in \mathbb{R}^n$, $\forall w \in \mathbb{R}^m$, $\forall a \in A$, $\forall \varepsilon > 0$.
iv. *$G^a : \mathbb{R}^n \to \mathbb{R}^m$ is a set-valued map, uniformly continuous with convex compact values on each set $\mathcal{B}_R(0)$, uniformly in $a \in A$.*
v. *$IntG^a \neq \phi$, $\forall x \in \mathbb{R}^n$, $\forall a \in A$.*

Then

$$\lim_{\varepsilon \to 0+} \sup_{a \in A} \left| \varepsilon \log \int_{\mathbb{R}^n} \int_{G^a(x)} e^{F_a^\varepsilon(x,w)/\varepsilon} dw dx - \sup_{x \in \mathbb{R}^n} \sup_{w \in G^a(x)} F_a(x,w) \right| = 0$$

PROOF. Write

$$\bar{F}_a^\varepsilon = \sup_{x \in \mathbb{R}^n} \sup_{w \in G^a(x)} F_a^\varepsilon(x,w)$$

$$\bar{F}_a = \sup_{x \in \mathbb{R}^n} \sup_{w \in G^a(x)} F_a(x,w).$$

Then our assumptions ensure that

$$\lim_{\varepsilon \to 0+} \sup_{a \in A} \left|\bar{F}_a^\varepsilon - \bar{F}_a\right| = 0.$$

For a given $\delta > 0$, define

$$\mathcal{B}_\delta^{a,\varepsilon} = \left\{x \in \mathbb{R}^n \mid \exists w \in G^a(x) \text{ s.t. } F_a^\varepsilon(x,w) > \bar{F}_a^\varepsilon - \delta\right\}.$$

Then assumption (iii) ensures that there exists $R > 0$ s.t.

$$\mathcal{B}_\delta^{a,\varepsilon} \subset \mathcal{B}_R(0).$$

Furthermore, by Berge's theorem [CV77], $L_G^a(\bar{\mathcal{B}}_R(0))$ is compact. Hence, $\exists \hat{R} > 0$ such that $L_G^a(\mathcal{B}_R(0)) \subset \mathcal{B}_{\hat{R}}(0)$.

By hypothesis (iii) on $\mathcal{B}_R(0) \times \mathcal{B}_{\hat{R}}(0)$ and using the uniform convergence of F_a^ε to F_a on $\bar{\mathcal{B}}_R(0) \times \bar{\mathcal{B}}_{\hat{R}}(0)$, $\exists r > 0$ such that

$$|x - x'| < \frac{r}{2} \text{ implies } |F_a^\varepsilon(x,w) - F_a^\varepsilon(x',w)| < \frac{\delta}{2}$$

for all $w \in \mathcal{B}_{\hat{R}(0)}$, $\forall x, x' \in \mathcal{B}_R(0)$, $a \in A$ and $\varepsilon > 0$ sufficiently small. Also,

$$|w - w'| < \frac{r}{2} \text{ implies } |F_a^\varepsilon(x,w) - F_a^\varepsilon(x,w')| < \frac{\delta}{2}$$

$\forall x \in \mathcal{B}_R(0)$, $\forall w, w' \in \mathcal{B}_{\hat{R}}(0)$, $a \in A$ and $\varepsilon > 0$ sufficiently small.
Pick a

$$x_a^\varepsilon \in \arg\max_{\mathbb{R}^n} \sup_{w \in G^a(x)} F_a^\varepsilon(x,w) \subset \mathcal{B}_R(0)$$

and

$$w_a^\varepsilon \in \arg \max_{w \in G^a(x_a^\varepsilon)} F_a^\varepsilon(x_a^\varepsilon, w) \subset \mathcal{B}_{\hat{R}}(0).$$

By compactness, $w_a^\varepsilon \in G^a(x_a^\varepsilon)$.
Now, let $\hat{\epsilon}$ be such that $0 < \hat{\epsilon} < \frac{r}{2}$, and define

$$\mathcal{W} \triangleq \{w \mid \ | w - w_a^\varepsilon | < \frac{r}{2} - \hat{\epsilon}\}.$$

Then, by the uniform continuity of G^a on $\mathcal{B}_{\hat{R}}(0)$, $\exists \hat{r}$ such that $\forall x$ with $| x_a^\varepsilon - x | < \hat{r}$

$$G^a(x) \bigcap \mathcal{W} \neq \phi$$

$\forall a \in A$.
Let $\bar{r} = \min\{\frac{r}{2}, \hat{r}\}$. Then, $\forall x \in \mathcal{B}_{\bar{r}}(x_a^\varepsilon)$ and for any $\bar{w}_x^a \in G^a(x) \bigcap \mathcal{W}$

$$\mathcal{B}_{\hat{\epsilon}}(\bar{w}_x^a) \bigcap G^a(x) \neq \phi$$

and for any $w \in \mathcal{B}_{\hat{\epsilon}}(\bar{w}_x^a) \bigcap G^a(x)$

$$| w - w_a^\varepsilon | < \frac{r}{2}.$$

Hence,

$$\mathcal{B}_{\bar{r}}(x_a^\varepsilon) \subset \mathcal{B}_\delta^{a,\varepsilon}, \quad \forall a \in A, \varepsilon > 0 \text{ sufficiently small.}$$

Now, let

$$\alpha_a^\varepsilon \triangleq \int_{\Re^n} \int_{w \in G^a(x)} \exp(F_a^\varepsilon(x, w)/\varepsilon) dw dx$$

For each $x \in \mathcal{B}_{\bar{r}}(x_a^\varepsilon)$ pick a $w_a^\varepsilon(x) \in G^a(x) \bigcap \mathcal{W}$. Then

$$\begin{aligned} \alpha_a^\varepsilon &\geq \int_{\mathcal{B}_{\bar{r}}(x_a^\varepsilon)} \int_{G^a(x) \bigcap \mathcal{B}_{\hat{\epsilon}}(w_a^\varepsilon(x))} \exp(F_a^\varepsilon(x, w)/\varepsilon) dw dx \\ &\geq \int_{\mathcal{B}_{\bar{r}}(x_a^\varepsilon)} \int_{G^a(x) \bigcap \mathcal{B}_{\hat{\epsilon}}(w_a^\varepsilon(x))} \exp\left(\frac{\bar{F}_a^\varepsilon - \delta}{\varepsilon}\right) dw dx \\ &\geq C_m \bar{r}^m a_{\hat{\epsilon}}^R \exp\left(\frac{\bar{F}_a^\varepsilon - \delta}{\varepsilon}\right). \end{aligned}$$

Which implies that

$$\begin{aligned} \varepsilon \log \alpha_a^\varepsilon &\geq \varepsilon \log C^m \bar{r}^m a_{\hat{\epsilon}}^R + \bar{F}_a^\varepsilon - \delta \\ &\geq \bar{F}_a - 3\delta \end{aligned}$$

for all $\varepsilon > 0$ sufficiently small and for all $a \in A$.
Next, for $R > 0$ write

$$\begin{aligned} \alpha_a^\varepsilon \triangleq &\int_{|x| \leq R} \int_{w \in G^a(x)} \exp\left(F_a^\varepsilon(x, w)/\varepsilon\right) dw dx + \\ &\int_{|x| \geq R} \int_{w \in G^a(x)} \exp\left(F_a^\varepsilon(x, w)/\varepsilon\right) dw dx \\ = &J + K. \end{aligned}$$

Note that

$$\varepsilon \log \alpha_a^\varepsilon = \varepsilon \log J + O(K/J)$$

for R sufficiently large.
Let

$$z \triangleq \begin{bmatrix} x \\ w \end{bmatrix}.$$

Then

$$\begin{aligned} K &\leq \int_{|z| \geq R} \exp\left(\frac{-\gamma_1 |z|^2 + \gamma_2}{\varepsilon}\right) dz \\ &\leq C_R \exp\left(\frac{C_1 - C_2 R^2}{\varepsilon}\right) \\ &\leq C_R \exp\left(-C'/\varepsilon\right) \end{aligned}$$

where $C_R, C_1, C_2 > 0$, and $C' > 0$ for R large enough.
Increase R if necessary to ensure that

$$\arg \max_{x \in \Re^n} \sup_{w \in G^a(x)} F_a^\varepsilon(x, w) \subset \mathcal{B}_R(0).$$

Then

$$\begin{aligned} \varepsilon \log J &\leq \varepsilon \log \int_{|x| \leq R} \int_{w \in G^a(x)} \exp\left(\bar{F}_a^\varepsilon/\varepsilon\right) dw dx \\ &\leq \varepsilon \log M_R \int_{|x| \leq R} \exp\left(\bar{F}_a^\varepsilon/\varepsilon\right) dx \\ &\leq \varepsilon \log C_n R^n M_R + \bar{F}_a^\varepsilon. \end{aligned}$$

Thus

$$\varepsilon \log \alpha_a^\varepsilon \leq \bar{F}_a + 3\delta$$

for all $\varepsilon > 0$ sufficiently small and for all $a \in A$. Thus

$$\sup_{a \in A} \left|\varepsilon \log \alpha_a^\varepsilon - \bar{F}_a\right| < 3\delta$$

for all $\varepsilon > 0$ sufficiently small. □

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Maintenance and Production Control of Manufacturing Systems with Setups

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Abstract. In this paper, we consider a one-machine, multiple-product manufacturing system with random breakdowns, preventive maintenance activities and repairs times. Each part-type requires processing for a specified length of time on the machine which may require both setup time and setup cost when switching from one type of product to another. Our objective is to find the preventive maintenance and production rates and a sequence of setups so as to minimize the total cost of setup and surplus. An approximate optimality condition is given in terms of approximated value functions based on the numerical approach. The numerical techniques, based on Kushner approach, are used to solve the optimal control problem. An example has been solved to illustrate the method. The resulting control policy is machine age dependent.

1. Introduction

The production planning problem for manufacturing systems subject to uncertainties such as demand fluctuations, machine failures, etc. has attracted the attention of numerous researchers. During the past 15 years, a number of methods have been reported for determining economic quantities for different products on a single machine or multiple machines. A class of scheduling policies is developed to stabilize the system in the sense that, in the long run, the required demand is met. The spirit of the classical approaches is more in keeping with the pioneering work of Kimemia and Gershwin [13] who study dynamic "close-loop" scheduling for systems with random machine failures. In the work of Kimemia and Gershwin [13], the system uncertainties are modeled by homogeneous finite state Markov processes. This line of work has been continued by Akella and Kumar [2], Bielecki and Kumar [3], Sharifnia [18] and Fleming et al. [10]. This concept was extended in works by Boukas (see [4] and [6]) where the author considered the fact that machine failure depends on its age.

The first studies dealing with the scheduling of the machines, which consists of their setup sequences, are based on a static and open-loop approach. In this approach, the issue of scheduling is usually formulated as an optimization problem

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