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The Combinatorial Basis of Uniformly Bounded Discrete Random Set Theory

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ABSTRACT

We provide a purely combinatorial view of the essential structure of uniformly bounded DRS theory without any reference to discrete probability, and highlight the two fundamental functionals arising out of this exposition. The key tool is Moebius inversion. We conclude that the study of uniformly bounded DRS theory is the study of Incidence functions on Boolean algebras of finite rank. Some useful results arise as a by-product of this investigation.

1. INTRODUCTION

The celebrated Choquet-Kendall-Matheron theorem [1, 2, 3, 4] for Random Closed Sets (RACS), states that a RACS X is completely characterized by its *capacity functional*, i.e., the collection of hitting probabilities over a sufficiently rich family of so-called *test sets*. RACS theory is a mature branch of theoretical and applied probability, whose scope is the study of set-valued random variables. There exist numerous references on the subject; e.g., cf. [3, 4, 2, 5, 6] for foundations, [7, 8, 9, 10, 11, 12, 13, 14, 15, 16] for statistical inference and applications, [17, 18, 19, 20] for a related statistical theory of shape, and [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40] for expectation, laws of large numbers, and related themes.

Our interest is in Uniformly Bounded Discrete Random Sets, or DRS for short, a special class of RACS defined on a finite lattice. A DRS may be thought of as a sampled version of an underlying RACS; cf. [41] for a rigorous algebraic analysis of a suitable sampling process, as well as a formal argument which establishes the usefulness of DRS theory. DRS's can be viewed as finite-alphabet random variables, taking values in a finite partially ordered set (poset). Thus, the only difference with ordinary finite-alphabet random variables is that the DRS alphabet naturally possesses only a partial order relation, instead of a total order relation. However, this simple deviation calls for a radical change in perspective.

One of the "wheels" of discrete probability and combinatorial theory is the celebrated principle of *inclusion-exclusion* (e.g., cf. the classic book of Feller [42]). Although this principle applies to numerous problems, it is often hard to make the connection, and realize that it indeed applies to a particular problem at hand. As noted by Rota [43] "It has often taken the combined efforts of many a combinatorial analyst over long periods to recognize an inclusion-exclusion pattern".

As it turned out, the inclusion-exclusion principle is the simplest but also the typical case of a very general principle of enumeration, regarding the inversion of indefinite sums ranging over an arbitrary poset. This principle is known as *Moebius inversion*, and it is the analog of the "fundamental Theorem of calculus" in the context of enumeration. Moebius inversion is one of the most pervasive "ground truths" in modern multi-dimensional signal processing; e.g., it is the basis of an important representation Theorem for Markov random fields. The first coherent instance of this principle is due to Weisner [44]. In its full generality it first appeared in Ward [45]. Excellent accounts can be found in the classic papers by Rota [43], and Crapo [46]; a more up-to-date exposition can be found in Aigner [47].

In [48] it was first observed that, for the special case of DRS's, a variant of Moebius inversion provides an elementary *constructive* proof of the Choquet-Kendall-Matheron Theorem. Here, constructive means that it allows one to compute the probabilities of *all* events of interest, based on the hitting probabilities. Published accounts can be found in [49, 50]. As a result, the use of the so-called *generating functional* (the complement of the capacity functional with respect to arithmetic unity) has been advocated in [48, 49, 50] for modeling and inference purposes. Goutsias [51] came up with a probabilistic proof of a similar result based on the inclusion-exclusion principle. He instead advocated the use of the so-called *cummulative distribution functional*, which measures the probability that a DRS X does not cover a given finite set. His argument was that the latter functional is a natural extension of the notion of the cummulative distribution function for ordinary random variables.

In this paper we revisit these approaches, demonstrate that both are, in fact, complementary incarnations of Moebius inversion, and, therefore, all the beauty of DRS theory has nothing to do with probability; the underlying tool is combinatorial theory. We also argue that, if our goal is the morphological processing of DRS's, then *both* the generating functional *and* the cummulative distribution functional are needed.

2. PRELIMINARIES

The following theorem is a cornerstone of enumeration and combinatorics. See [47] for a general proof.

Theorem 1 (Moebius inversion) *Let B be a finite set (i.e., $|B| < \infty$), and $\Sigma(B)$ its power set. $(\Sigma(B), \subseteq)$ is a complete lattice with unit element B and zero element $\emptyset$. $(\Sigma(B), \subseteq)$ is isomorphic to the Boolean Algebra of (finite) rank $|B|$. Let p be a real-valued functional on $\Sigma(B)$. Define the lower and upper sum functionals, q, and r, respectively, by*

$$q(A) \triangleq \sum_{S \subseteq A} p(S), \ \forall A \in \Sigma(B)$$

$$r(A) \triangleq \sum_{S \supseteq A} p(S), \ \forall A \in \Sigma(B)$$

Then, $\forall S \in \Sigma(B)$ (inversion from below)

$$p(S) = \sum_{A \subseteq S} (-1)^{|A|} q(S \cap A^c) = \sum_{A \subseteq S} (-1)^{|S|-|A|} q(A)$$

and, $\forall S \in \Sigma(B)$ (inversion from above)

$$p(S) = \sum_{A \supseteq S} (-1)^{|A^c|} r(S \cup A^c) = \sum_{A \supseteq S} (-1)^{|A|-|S|} r(A)$$

where $A^c = B - A$, $\forall A \in \Sigma(B)$.

A stand-alone proof involves a convenient change of variables, followed by the application of an enumeration Lemma.

We will need the following technical Lemma. Strangely enough, we haven't been able to locate it in the classic references [43, 46, 52], or in the relatively up-to-date book of Aigner [47]. A proof can be constructed using Moebius inversion from above. We skip it here due to space limitations.

Lemma 1

$$q(A) = \sum_{S \subseteq A^c} (-1)^{|S|} r(S) = \sum_{S \supseteq A} (-1)^{|S^c|} r(S^c), \ \forall A \in \Sigma(B)$$

and

$$r(A) = \sum_{S \subseteq A} (-1)^{|S|} q(S^c) = \sum_{S \supseteq A^c} (-1)^{|S^c|} q(S), \ \forall A \in \Sigma(B)$$

Consider a mapping $\phi : \Sigma(B) \mapsto \Sigma(B)$. ϕ is called an *erosion* if it distributes over intersection. ϕ is called a *dilation* if it distributes over union. A pair (ϵ, δ) of mappings from $\Sigma(B)$ to itself is called an *adjunction* on $\Sigma(B)$ if[1]

$$\delta(A) \subseteq S \Longleftrightarrow A \subseteq \epsilon(S), \ \forall A \in \Sigma(B), \forall S \in \Sigma(B)$$

[1]The definitions and properties given here are sufficient for our purposes; see [53] for a general treatment.

If (ϵ, δ) is an adjunction, then ϵ is necessarily an erosion (i.e., it distributes over intersection), and δ is a dilation (i.e., it distributes over union). For any given erosion, ϵ, there exists a unique dilation, δ, dubbed the *right adjoint* of ϵ, such that the pair (ϵ, δ) is an adjunction. This δ is given by

$$\delta(A) = \cap \{S \in \Sigma(B) \mid A \subseteq \epsilon(S)\}$$

Similarly, for any given dilation, δ, there exists a unique erosion, ϵ, dubbed the *left adjoint* of δ, such that the pair (ϵ, δ) is an adjunction. This ϵ is given by

$$\epsilon(S) = \cup \{A \in \Sigma(B) \mid \delta(A) \subseteq S\}$$

If (ϵ, δ) is an adjunction, then ϵ and δ are *adjoint* to each other. A thorough treatment of adjunctions can be found in [53]. Related material can also be found in [54].

We have the following elementary Lemma. The proof follows along the lines of [53, pp. 85-86].

Lemma 2 *Any dilation, δ, can be represented as*

$$\delta(A) = \cup_{z \in A} S(z)$$

Given $\delta(A) = \cup_{z \in A} S(z)$ define

$$\epsilon(A) \triangleq \{z \in B \mid S(z) \subseteq A\}$$

It can be proven [53, pp. 85-86], that (ϵ, δ) is an adjunction. Furthermore,

Lemma 3 *Any erosion, ϵ, can be represented as*

$$\epsilon(A) = \{z \in B \mid S(z) \subseteq A\}$$

The proof follows by uniqueness of left adjoint. We will need the following Lemma. The proof is elementary.

Lemma 4 *Let (ϵ, δ) be an adjunction. Define*

$$q_\delta(A) \triangleq \sum_{\delta(S) \subseteq A} p(S)$$

and

$$r_\epsilon(A) \triangleq \sum_{\epsilon(S) \supseteq A} p(S)$$

Then

$$q_\delta(A) = q(\epsilon(A)), \quad \forall A \in \Sigma(B)$$

and

$$r_\epsilon(A) = r(\delta(A)), \quad \forall A \in \Sigma(B)$$

3. CONNECTION WITH UNIFORMLY BOUNDED DRS THEORY

Let us now make the connection with DRS theory. A DRS X is simply a measurable mapping from some abstract probability space to the measurable space $(\Sigma(B), \Sigma(\Sigma(B)))$. As such, it induces a unique probability measure on $\Sigma(\Sigma(B))$. Denote this by $P_X(\cdot)$. Let $p_X(\cdot)$ denote the restriction of $P_X(\cdot)$ to the atoms, i.e., the elements of $\Sigma(B)$. This is the **probability mass function** of the DRS X. Define the **capacity functional, generating functional, covering functional, and cummulative distribution functional**, $T_X(\cdot), Q_X(\cdot), R_X(\cdot), F_X(\cdot)$, respectively, by

$$T_X(A) \triangleq P_X(X \cap A \neq \emptyset)$$

$$Q_X(A) \triangleq P_X(X \cap A = \emptyset) = 1 - T_X(A) = \sum_{S \subseteq A^c} p_X(S)$$

$$R_X(A) \triangleq P_X(X \supseteq A) = \sum_{S \supseteq A} p_X(S)$$

$$F_X(A) \triangleq 1 - R_X(A)$$

Identify p_X with the functional p of Theorem 1. Then, it becomes clear that

$$Q_X(A) = q(A^c)$$

and

$$R_X(A) = r(A)$$

where q, r are the lower, and upper sum functionals, respectively, of Theorem 1. It then follows from Theorem 1, and Lemma 1, that $\forall S \in \Sigma(B)$

$$p_X(S) = \sum_{A \subseteq S} (-1)^{|A|} Q_X(S^c \cup A) = \sum_{A \subseteq S} (-1)^{|S|-|A|} Q_X(A^c)$$

and

$$p_X(S) = \sum_{A \supseteq S} (-1)^{|A^c|} R_X(S \cup A^c) = \sum_{A \supseteq S} (-1)^{|A|-|S|} R_X(A)$$

Furthermore, $\forall A \in \Sigma(B)$

$$Q_X(A) = \sum_{S \subseteq A} (-1)^{|S|} R_X(S) = \sum_{S \supseteq A^c} (-1)^{|S^c|} R_X(S^c)$$

and,

$$R_X(A) = \sum_{S \subseteq A} (-1)^{|S|} Q_X(S) = \sum_{S \supseteq A^c} (-1)^{|S^c|} Q_X(S^c)$$

Since $Q_X(A) = 1 - T_X(A)$ and $F_X(A) = 1 - R_X(A)$, $\forall A \in \Sigma(B)$, we now have identities which relate all five functionals T_X, Q_X, R_X, F_X, and p_X. As a trivial corollary, any one of these functionals is a sufficient and constructive specification of X.

4. CONNECTION WITH MATHEMATICAL MORPHOLOGY

Mathematical morphology is an important quantitative shape analysis tool in image processing. Its foundations were laid down by Matheron [3, 4], Serra [10, 11], and collaborators, during the late '60's to early '80's. Since then, the theory has been generalized by its founders, as well as several other researchers; an excellent recent treatment which unifies several seemingly distinct approaches within a purely algebraic framework can be found in [53].

Morphological Image Operators [53] are compositions of two classes of elementary building blocks, namely erosions and dilations. Several types of erosions and dilations can be defined; the definition given in section 2. is the most general one within our framework.

Let (ϵ, δ) be an adjunction. Identify p_X with the functional p of Theorem 1. Then, as before, $Q_X(A) = q(A^c)$ and $R_X(A) = r(A)$. We further observe that $Q_{\delta(X)}(A) = q_\delta(A^c)$, and $R_{\epsilon(X)}(A) = r_\epsilon(A)$, where q_δ, and r_ϵ have been defined in Lemma 4. By applying this latter Lemma, we conclude

$$Q_{\delta(X)}(A) = Q_X((\epsilon(A^c))^c), \quad \forall A \in \Sigma(B)$$

and

$$R_{\epsilon(X)}(A) = R_X(\delta(A), \quad \forall A \in \Sigma(B)$$

A probabilistic proof of the first result for the special case of translation-invariant operators has appeared in [48]; similarly, a probabilistic proof of the second result for the special case of translation-invariant operators has appeared in [51].

In the previous section we have concluded that either one of the functionals T_X, Q_X, R_X, F_X, or p_X, is a sufficient and constructive specification of X. From the latter two identities we now conclude that Q_X (or, equivalently, $T_X = 1 - Q_X$) is the most convenient specification if our interest is in processing X via an operator which distributes over union (i.e., a dilation), whereas R_X (or, equivalently, $F_X = 1 - R_X$) is the most convenient specification if our interest is in processing X via an operator which distributes over intersection (i.e., an erosion). This picture is depicted in figure 1. One may obviously substitute T_X for Q_X,

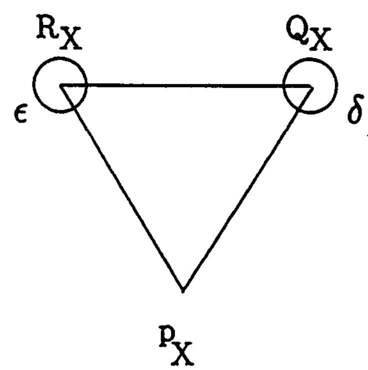

Figure 1. The two fundamental functionals of DRS theory

and/or F_X for R_X. The choice of Q_X versus R_X depends solely on whether one will apply a dilation, or erosion, respectively, on X. However, in morphological shape analysis and synthesis we sequentially process X using a variety of erosions and dilations. This means that one is forced to use *both* specifications, i.e., move between the two in anticipation of the next operation in line. This movement is made possible by Lemma 1, which is essentially another incarnation of Moebius inversion. As such, it involves a combinatorial computational cost. This is the single most important barrier in applications.

5. CONCLUSIONS

We have argued that all the essential structure of DRS theory has nothing to do with probability: it is purely due to a synergy of underlying combinatorics. We may then conclude that the study of DRS theory is the study of so-called *Incidence functions* (in the sense of [47]) on Boolean algebras of finite rank.

DRS theory inherits two fundamental functionals from combinatorial theory. There is nothing "magic" about the choice of a "working" functional specification of a DRS X. It all depends on the kind of operations one is interested in. In general, one is forced to use *both* functionals, and this brings *complexity* into the picture. In our view, this is the single most important obstacle in applications.

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