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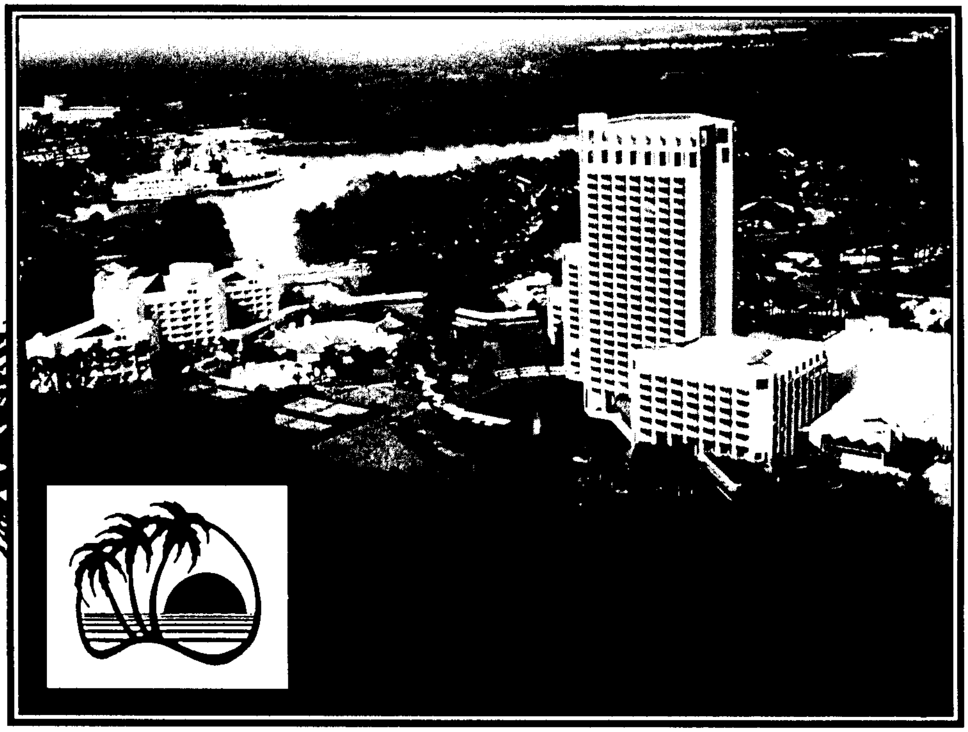

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NONLINEAR H_∞ CONTROL*

J.S. Baras[1] M.R. James[2]

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[1] Martin Marietta Chair in Systems Engineering, Department of Electrical Engineering and Institute for Systems Research, University of Maryland, College Park, MD 20742, USA.

[2] Department of Systems Engineering, Australian National University, Canberra, ACT 0200, Australia.

Abstract

This paper discusses various aspects of recent developments concerning the robust H_∞ problem for continuous-time nonlinear systems. In the state feedback case, the problem can be solved in the framework of dissipative systems, and the associated storage functions solve the PDI in the viscosity sense. The main problem is then one of controller synthesis. In the output feedback case, the information state concept provides the appropriate framework. The solution is expressed in terms of a PDE for the information state (filter), and an infinite dimensional PDE or dissipation-type PDI for the value function (control). The attendant mathematical issues are discussed.

1. Introduction

The robust H_∞ control problem for nonlinear systems has attracted considerable interest in the last few years. This interest follows the complete solution to the linear H_∞ problem obtained over the last decade or so, involving a filter-type Riccati equation, a control-type Riccati equation, and a coupling condition. In addition, the H_∞ problem has sparked new interest in risk-sensitive stochastic control, differential games, and dissipative systems.

Most approaches to the nonlinear H_∞ problem have sought to generalize in some way the linear solution, e.g., by replacing the Riccati equations with Hamilton-Jacobi-Bellman (HJB) or Hamilton-Jacobi-Isaacs (HJI) equations, or inequalities [1], [3], [9], [16], [17]. This approach was successful in the state feedback case, although the problem of controller synthesis remains an issue. In the output feedback case, this approach did not work as expected. Indeed, a new framework was required [11], [12], [13]. The key to the output feedback case was the use of an appropriate *information state*, a quantity providing the correct notion of "state". In [13], both necessary and sufficient conditions were given in terms of (discrete-time analogs of) a PDE for the information state (filter), and an *infinite dimensional* partial differential inequality (PDI) for a value function (control). For a special class of systems, the information state turns out to be finite-dimensional, see [15].

In this paper, we discuss in heuristic terms the solution of the nonlinear H_∞ problem in continuous-time (complete details for the discrete-time case are provided in [13], and the continuous-time case is analysed in detail in [14]). Our objective in this paper is to present the basic equations and to explain some of the attendant mathematical issues which arise, in both the state and output feedback cases. We begin in §2 by stating, in quite general terms, the problem to be solved. Then in §3 the solution to state feedback case is reviewed, and a number of technical issues relating to the general lack of smoothness of value functions are discussed. The output feedback case is discussed in §4, where a new infinite dimensional PDI is introduced.

2. Problem Formulation

We consider continuous-time nonlinear systems Σ described by the state space equations of the general form

$$\begin{cases} \dot{x}(t) &= b(x(t),u(t),w(t)), \\ z(t) &= l(x(t),u(t),w(t)), \\ y(t) &= h(x(t),u(t),w(t)). \end{cases} \quad (2.1)$$

Here, $x(t) \in \mathbf{R}^n$ denotes the state of the system, and is not in general directly measurable; instead an ouput quantity $y(t) \in \mathbf{R}^p$ is observed. The additional output quantity $z(t) \in \mathbf{R}^q$ is a performance measure, depending on the particular problem at hand. The control input is $u(t) \in U \subset \mathbf{R}^m$, and $w(t) \in \mathbf{R}^r$ is a disturbance input. For instance, w could be due to modelling errors, sensor noise, etc. The system behavior is determined by the functions $b : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^r \to \mathbf{R}^n$, $l : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^r \to \mathbf{R}^q$, $h : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^r \to \mathbf{R}^p$, assumed sufficiently smooth. It is assumed that the origin is an equilibrium for the system (2.1): $b(0,0,0) = 0$, $l(0,0,0) = 0$, and $h(0,0,0) = 0$.

The *output feedback robust H_∞ control problem* is: given $\gamma > 0$, find a controller $\mathbf{u} = \mathbf{u}(y(\cdot))$, responsive

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only to the observed output y, such that the resulting closed loop system $\Sigma^{\mathbf{u}}$ achieves the following two goals;

(i) $\Sigma^{\mathbf{u}}$ is asymptotically stable when no disturbances are present, and

(ii) $\Sigma^{\mathbf{u}}$ is *finite gain*, i.e., for each initial condition $x_0 \in \mathbf{R}^n$ the input–output map $\Sigma^{\mathbf{u}}_{x_0}$ relating w to z is finite gain, which means that there exists a finite quantity $\beta^{\mathbf{u}}(x_0)$ such that

$$\begin{cases} \int_0^T |z(t)|^2\, dt \;\le\; \gamma^2 \int_0^T |w(t)|^2\, dt + \beta^{\mathbf{u}}(x_0) \\ \text{for all } \; w \in L_2([0,T], \mathbf{R}^r) \text{ and all } T \ge 0. \end{cases} \tag{2.2}$$

Since $x_0 = 0$ is an equilibrium, we also require that $\beta^{\mathbf{u}}(0) = 0$.

3. The State Feedback Case

The state feedback problem is much simpler than the output feedback problem posed in the preceding section. Nevertheless, in continuous-time there are still some important technical issues not fully addressed in the literature.

A state feedback controller can use causal information concerning the state, and actually it is enough to consider controllers $\mathbf{u}$ depending only on the state at time t: $u(t) = \mathbf{u}(x(t))$. As is well known, the H_∞ problem can be posed and solved using differential game methods, see [4]. In the nonlinear case, all the usual difficulties encountered in optimal control and game theory relating to lack of smoothness of value functions and possibly discontinuous optimal feedback control policies, if they indeed exist, are apparent, as we shall see.

The state feedback solution can be concisely discussed in terms of dissipative systems [2], [8], [10], [18]. Let $\mathbf{u}$ be a given state feedback controller, with closed loop system $\Sigma^{\mathbf{u}}$. From the theory of dissipative systems, $\Sigma^{\mathbf{u}}$ is finite gain (in the sense of §2) if and only if $\Sigma^{\mathbf{u}}$ is *finite gain dissipative*, i.e. there exists a function $V(x)$ (called a *storage function*) such that $V(0) = 0$, $V(x) \ge 0$, and

$$V(x) \ge \sup_{T,w}\{V(x(T)) - \int_0^T (\gamma^2 |w(s)|^2 - |z(s)|^2) ds \;:\; x(0) = x\}. \tag{3.1}$$

The relation (3.1) is known as the *dissipation inequality*, and is equivalent to the PDI

$$\sup_w \{\nabla_x V \cdot b(x, \mathbf{u}(x), w) - \gamma^2 |w|^2 + |\ell(x, \mathbf{u}(x), w)|^2\} \le 0 \text{ in } \mathbf{R}^n. \tag{3.2}$$

The storage function $V(x)$ need not be smooth (e.g. C^1), and so (3.2) must be interpreted in a generalized sense [2], [10]. In [10], $V(x)$ is assumed merely locally bounded, and the PDI (3.2) completely characterizes such storage functions using viscosity solution methods. Consequently, $\Sigma^{\mathbf{u}}$ is finite gain if and only if there exists a function $V(x)$ satisfying $V(0) = 0$, $V(x) \ge 0$, and the PDI (3.2) in the viscosity sense.

The above considerations apply to a fixed state feedback controller $\mathbf{u}$. (We have tacitly assumed that $\mathbf{u}$ enjoys sufficient regularity to ensure that solutions to the ODE in (2.1) exist and are unique, etc.) If the H_∞ problem is solvable by $\mathbf{u}$, then the PDI holds for some storage function V, and consequently, after minimizing with respect to u, the PDI

$$\inf_u \sup_w \{\nabla_x V \cdot b(x, u, w) - \gamma^2 |w|^2 + |\ell(x, u, w)|^2\} \le 0 \text{ in } \mathbf{R}^n \tag{3.3}$$

is satisfied by V. Inequality (3.3) is the fundamental PDI for the state feedback problem, and is closely related to the HJI equation of arising in game theory. The necessity of (3.3) is evident.

Sufficiency results can also be obtained, depending on the level of verification theorem available, [5], [7]. Indeed, if there exists a C^1 solution $V(x)$ to (3.3) and if $\mathbf{u}^*(x)$ attains the minimum in (3.3) at each x (and satisfies some regularity conditions), then $\mathbf{u}^*$ solves the state feedback H_∞ problem, assuming suitable controllability and observability conditions are met (to ensure stability). Because solutions of the PDI (3.3) are generally not smooth, then more general verification theorems need to be employed. This is the problem of *controller synthesis.*

The difficulties regarding lack of smooth solutions to the PDI (3.3) can be circumvented in part by working locally near the equilibrium. With additional assumptions, it is possible to show that smooth solutions exists in neighborhood $\mathcal{N}$ of 0, and to solve the H_∞ problem locally in $\mathcal{N}$. This essentially makes use of the classical method of characteristics for first-order equations, and is expressed in geometrical language in [16], [17]. Indeed, if the H_∞ problem for the linearized system is solvable, then the nonlinear problem is solvable locally.

4. The Output Feedback Case

To solve the output feedback problem, one would like to use the game theoretic/dissipative systems methods used in the state feedback case. However, this is difficult without the definition of a suitable notion of "state". The information state [11], [12], [13] provides such a notion.

Consider the following finite horizon minimax differential game. The problem is to minimize, over all (admissible) output feedback controllers $\mathbf{u} \in \mathcal{O}$, the cost function

$$J_{\bar{p},T}(\mathbf{u}) = \sup_{w,x_0}\{\bar{p}(x(0)) + \int_0^T(|z(s)|^2 - \gamma^2|w|^2)\,ds$$
$$+\Phi(x(T))\}, \tag{4.1}$$

where $\bar{p} \in \mathcal{E}$, a suitable (infinite dimensional) space of extended real-valued functions, and Φ is smooth. Define the function $\delta_x \in \mathcal{E}$ by

$$\delta_x(\xi) = \begin{cases} 0 & \text{if } \xi = x, \\ -\infty & \text{if } \xi \neq x. \end{cases}$$

The finiteness of each $J_{\delta_{x_0},T}(u)$, $x_0 \in \mathbf{R}^n$, is equivalent to the finite gain property (2.2) (when $\Phi = 0$). The key to solving this output feedback game is to replace it with an equivalent state feedback game involving a suitable information state.

To this end, fix an output path $y \in L_2([0,T],\mathbf{R}^p)$ and define the *information state* $p_t \in \mathcal{E}$ by

$$p_t(x) = \sup_{w,x_0}\{\bar{p}(x(0)) + \int_0^t(|z(s)|^2 - \gamma^2|w|^2$$
$$+\delta_{y(s)}(h(x(s),u(s),w(s)))\,ds \;:\; x(t) = x\}. \tag{4.2}$$

In words, this quantity describes the worst-case performance up to time t using the controller $\mathbf{u}$ which is consistent with the observed output and the constraint $x(t) = x$. It summarizes the observed information in a way which is suitable for fulfilling the control objective. The information state evolves according to the dynamics

$$\dot{p}_t = F(p_t, u(t), y(t)), \quad p_0 = \bar{p}, \tag{4.3}$$

where $F(p,u,y)$ is the nonlinear operator

$$F(p,u,y) = \sup_w[-\nabla_x p \cdot b(\cdot,u,w) + |\ell(\cdot,u,w)|^2$$
$$-\gamma^2|w|^2 + \delta_y(h(\cdot,u,w))]. \tag{4.4}$$

Equation (4.3) is a first-order nonlinear PDE in $\mathbf{R}^n$. The cost function has the following representation purely in terms of the information state:

$$J_{\bar{p},T}(\mathbf{u}) = \sup_{y\in L_2([0,T],\mathbf{R}^p)}\{(p_T,\Phi) \;:\; p_0 = \bar{p}\}, \tag{4.5}$$

where $(p,\Phi) = \sup_{x\in\mathbf{R}^n}(p(x) + \Phi(x))$ is the "sup-pairing ", [11].

The problem can now be solved using dynamic programming methods, with cost given by the RHS of (4.5) and dynamics (4.3). The value function is

$$W(p,t) = \inf_{\mathbf{u}\in\mathcal{O}} \sup_{y\in L_2([0,T],\mathbf{R}^p)}\{(p_T,\Phi) \;:\; p_t = p\}, \tag{4.6}$$

and the corresponding dynamic programming equation is (assuming min and max can be interchanged)

$$\begin{aligned} \tfrac{\partial W}{\partial t} + \inf_{u\in U}\sup_{y\in\mathbf{R}^p}\nabla_p W\cdot F(p,u,y) &= 0, \\ W(p,T) &= (p,\Phi). \end{aligned} \tag{4.7}$$

The optimal minmax controller $\mathbf{u}^*(p,t)$ is obtained by finding the control value which attains the minimum in (4.7), assuming validity of a verification theorem. Note that this controller is an information state feedback controller, and since p_t is a causal function of y, it is also an output feedback controller.

The mathematical issues here involve the correct definition of solutions to infinite dimensional nonlinear PDEs of the type (4.7), and related matters. Equations of the type (4.7) are new [12], and much work remains to be done. Complete details are available in the discrete-time case [11], [13], and for a class of continuous-time systems in [14]. An interesting feature of the function W is that it has a nontrivial domain, even when specialized to the linear case (see the example in [15]).

The infinite horizon problem is treated by letting the above time horizon tend to infinity. The resulting PDE will be a stationary one, and motivated by the inequalities arising in dissipative systems, it is enough to consider the PDI

$$\inf_{u\in U}\sup_{y\in\mathbf{R}^p}\nabla_p W\cdot F(p,u,y) \;\leq\; 0 \text{ in } \mathcal{E}. \tag{4.8}$$

This is the fundamental PDI for the output feedback problem. Note that it is *infinite dimensional.* Additionally, $W(p)$ must satisfy $W(p) \geq (p,0)$ and $W(-\beta) = 0$, for some $\beta \geq 0$ (as in (2.2)).

Necessary and sufficient conditions for the solvability of the output feedback H_∞ problem can be expressed in terms of the PDE (4.3) for the information state (filter), and the infinite dimensional PDI (4.8) (control). The coupling condition is that $p_t \in \mathrm{dom}W$, $t \geq 0$. Of course, suitable controllability/observability conditions are needed, [13]. Assuming the validity of a verification theorem, the solution to the H_∞ problem is the information state feedback controller $\mathbf{u}^*(p)$ defined by finding the control which attains the minimum in (4.8) for each p. As in the state feedback case, the problem of controller synthesis is a crucial, and difficult, one.

In summary, we have presented the fundamental equations for the solution of the robust H_∞ problem for continuous-time nonlinear systems. The equations are natural analogs of the corresponding discrete-time equations [13], but involve considerably more technical difficulties, see [14].

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