

# Convergence of Kohonen's Learning Vector Quantization

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## Abstract

Kohonen's Learning Vector Quantization is a nonparametric classification scheme which classifies observations by comparing them to  $k$  templates called Voronoi vectors. The locations of these vectors are determined from past labeled data through a learning algorithm. When learning is complete, the class of a new observation is the same as the class of the closest Voronoi vector. Hence LVQ is similar to nearest neighbors, except that instead of all of the past observations being searched only the  $k$  Voronoi vectors are searched.

In this paper, we show that the LVQ learning algorithm converges to locally asymptotic stable equilibria of an ordinary differential equation. It is shown that the learning algorithm performs stochastic approximation. Convergence of the vectors is guaranteed under the appropriate conditions on the underlying statistics of the classification problem. We also present a modification to the learning algorithm which results in more robust convergence.

## 1.1 Learning Vector Quantization

The LVQ algorithm is now described. Let  $\{(x_i, d_{x_i})\}_{i=1}^N$  be the training data or past observation set. This means that  $x_i$  is observed when pattern  $d_{x_i}$  is in effect. Let  $\theta_j$  be a Voronoi vector and let  $\Theta = \{\theta_1, \dots, \theta_k\}$ . We assume that there are many more observations than Voronoi vectors (Duda & Hart [1973]). Once the Voronoi vectors are initialized, training proceeds by taking a sample  $(x_j, d_{x_j})$  from the training set, finding the closest Voronoi vector and adjusting its value according to equations (1) and (2). After several passes through the data, the Voronoi vectors converge and training is complete.

Suppose  $\theta_c$  is the closest vector then  $\theta_c$  is adjusted as follows:

$$\theta_c(n+1) = \theta_c(n) + \alpha_n (x_j - \theta_c(n)) \quad (1)$$

if  $d_{\theta_c} = d_{x_j}$  and

$$\theta_c(n+1) = \theta_c(n) - \alpha_n (x_j - \theta_c(n)) \quad (2)$$

if  $d_{\theta_c} \neq d_{x_j}$ . The other Voronoi vectors are not modified.

This update has the effect that if  $x_j$  and  $\theta_c$  have the same decision then  $\theta_c$  is moved closer to  $x_j$ , however if they have different decisions then  $\theta_c$  is moved away from  $x_j$ . The constants  $\{\alpha_n\}$  are positive and decreasing, e.g.,  $\alpha_n = 1/n$ .

## 1.2 Convergence of the Learning Algorithm

The LVQ algorithm has the general form

$$\theta_i(n+1) = \theta_i(n) + \alpha_n \gamma(d_{x_n}, d_{\theta_i(n)}, x_n, \Theta_n) (x_n - \theta_i(n)) \quad (3)$$

where  $x_n$  is the currently chosen past observation. The function  $\gamma$  determines whether there is an update and what its sign should be. It is given by

$$\gamma(d_{x_n}, d_{\theta_i}, x_n, \Theta_n) = 1_{\{x_n \in V_{\theta_i}\}} (1_{\{d_{x_n} = d_{\theta_i}\}} - 1_{\{d_{x_n} \neq d_{\theta_i}\}}). \quad (4)$$

Here  $1_{\Omega}$  represents the indicator function and  $V_{\theta_j}$  represents the set of points closest to  $\theta_j$ .

The update in (3) is a stochastic approximation algorithm (Benveniste, Metivier & Priouret [1987]). It has the form

$$\Theta_{n+1} = \Theta_n + \alpha_n H(\Theta_n, z_n) \quad (5)$$

where  $\Theta$  is the vector with components  $\theta_i$ ;  $H(\Theta, z)$  is the vector with components defined in the obvious manner from (3) and  $z_n$  is the random pair consisting of the observation and the associated *true* pattern number. If the appropriate conditions are satisfied by  $\alpha_n$ ,  $H$ , and  $z_n$ , then  $\Theta_n$  approaches the solution of

$$\frac{d}{dt} \bar{\Theta}(t) = h(\bar{\Theta}(t)) \quad (6)$$

for the appropriate choice of  $h(\Theta)$ .

Let  $p_i(x)$  represent the pattern density for pattern  $i$  and let  $\pi_i$  represent its prior. Suppose there are  $\ell$  patterns. It can be shown (Kohonen [1986]) that

$$h_i(\Theta) = \int_{V_{\theta_i}} (x - \Theta_i) p_i(x) \pi_i dx - \sum_{\substack{j=1 \\ j \neq \theta_i}}^{\ell} \int_{V_{\theta_j}} (x - \Theta_j) p_j(x) \pi_j dx \quad (7)$$

The following hypotheses are assumed:

**[H.1]**  $\{\alpha_n\}$  is a nonincreasing sequence of positive reals such that  $\sum_n \alpha_n = \infty$ ,  $\sum_n \alpha_n^\lambda < \infty$ .

**[H.2]** Given  $d_{x_n}$ ,  $x_n$  are independent and distributed according to  $p_{d_{x_n}}(x)$ .

**[H.3]** The pattern densities,  $p_i(x)$ , are continuous.

With these assumptions it is possible, using techniques from (Benveniste, Metivier & Priouret [1987]) or (Kushner & Clark [1978]), to prove the following theorem.

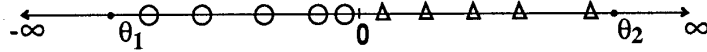


Figure 1: A possible distribution of observations and two Voronoi vectors.

**Theorem 1** Assume that [H.1]–[H.3] hold. Let  $\bar{\Theta}^*$  be a locally asymptotic stable equilibrium point of (6) with domain of attraction  $D^*$ . Let  $Q$  be a compact subset of  $D^*$ . If  $\Theta_n \in Q$  for infinitely many  $n$  then

$$\lim_{n \rightarrow \infty} \Theta_n = \bar{\Theta}^* \quad a.s. \quad (8)$$

**Proof:** (see (LaVigna [1989]))

Hence if the initial locations and decisions of the Voronoi vectors are close to a locally asymptotic stable equilibrium of (6) and if they do not move too much then the vectors converge.

### 1.3 Modified LVQ Algorithm

The convergence result above requires that the initial conditions are close to the stable points of (6) in order for the algorithm to converge. In this section we present a modification to the LVQ algorithm which increases the number of stable equilibrium for equation (6) and hence increases the chances of convergence. First we present a simple example which emphasizes a defect of LVQ and suggests an appropriate modification to the algorithm.

Let  $\circ$  represent an observation from pattern 2 and let  $\triangle$  represent an observation from pattern 1. We assume that the observations are scalar. Figure 1 shows a possible distribution of observations. Suppose there are two Voronoi vectors  $\theta_1$  and  $\theta_2$  with decisions 1 and 2, respectively, initialized as shown in Figure 1. At each update of the LVQ algorithm, a point is picked at random from the observation set and the closest Voronoi vector is modified. We see that during this update,  $\theta_2(n)$  is pushed towards  $\infty$  and  $\theta_1(n)$  is pushed towards  $-\infty$ , hence the Voronoi vectors do not converge.

This divergence happens because the decisions of the Voronoi vectors do not agree with the majority vote of the observations closest to each vector. As a result, the Voronoi vectors are pushed away from the origin. It is interesting to note that this phenomena occurs even though the observation data is bounded. This divergence is a result of the bi-directional learning algorithm.

The example above points out the fact that if the decision associated with a Voronoi vector does not agree with the majority vote of the observations closest to that vector then it is possible for the vector to diverge. A solution to this problem is to modify the decisions of

all the Voronoi vectors after each iteration of the learning algorithm. This modification will ensure that their decisions correspond to the majority vote and facilitate further theoretical analysis.

In practice the decisions would only be modified during the initial iterations of the learning algorithm since this is when  $\alpha_n$  is large and the Voronoi vectors are moving around significantly. With this modification it is possible to show convergence of the Voronoi vectors for a larger set of initial conditions (LaVigna [1989]). It is also possible to show convergence of the classification error to the Bayes optimal as the number of Voronoi vectors become large (LaVigna [1989]).

## 1.4 Conclusions

We have shown convergence of the Voronoi vectors in the LVQ algorithm. We have also presented the majority vote modification of the LVQ algorithm. This modification prevents divergence of the Voronoi vectors and results in convergence for a larger set of initial conditions. In addition, with this modification it is possible to show that as the appropriate parameters go to infinity the decision regions associated with the modified LVQ algorithm approach the Bayesian optimal (LaVigna [1989]). We are currently using these techniques to analyse the convergence of LVQ2.

## 1.5 Acknowledgements

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