The Conditional Adjoint Process

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Summary

The adjoint process and minimum principle for a partially observed diffusion can be obtained by differentiating the statement that a control u^* is optimal. Using stochastic flows the variation in the cost resulting from a change in an optimal control can be computed explicitly. The technical difficulty is to justify the differentiation.

1. INTRODUCTION.

Using stochastic flows we calculate below the change in the cost due to a 'strong' variation of an optimal control. Differentiating this quantity enables us to identify the adjoint, or co-state variable, and give a partially observed minimum principle. If the drift coefficient is differentiable in the control variable the related result of Bensoussan [2] follows from our theorem. Full details will appear in [1]. The method appears simpler than that employed in Haussman [4].

[1]Partially supported by US Army Contract DAAL03-86-C-0014 and by NSF Grant CDR-85-00108.
[2]Partially supported by the Natural Sciences and Engineering Research Council of Canada under grant A-7964 and the Air Force Office of Scientific Research, United States Air Force, under grant AFOSR-86-0332.
[3]Partially supported by the Natural Sciences and Engineering Research Council of Canada under grant A-7964.

2. DYNAMICAL EQUATIONS.

Suppose the state of a stochastic system is described by the equation

$$d\xi_t = f(t,\xi_t,u)dt + g(t,\xi_t)dw_t,$$
$$\xi_t \in R^d, \qquad \xi_0 = x_0, \qquad 0 \le t \le T. \tag{2.1}$$

The control variable u will take values in a compact subset U of some Euclidean space R^k. We shall assume

A_1: $x_0 \in R^d$ is given.

A_2: $f : [0,T] \times R^d \times U \to R^d$ is Borel measurable, continuous in u for each (t,x), continuously differentiable in x for each (t,u) and

$$(1+|x|)^{-1}|f(t,x,u)| + |f_x(t,x,u)| \le K_1.$$

A_3: $g : [0,T]\times R^d \to R^d \otimes R^n$ is a matrix valued function, Borel measurable, continuously differentiable in x, and for some K_2:

$$|g(t,x)| + |g_x(t,x)| \le K_2.$$

The observation process is defined by

$$dy_t = h(\xi_t)dt + d\nu_t \tag{2.2}$$
$$y_t \in R^m, \qquad y_0 = 0, \qquad 0 \le t \le T.$$

In (2.1) and (2.2) $w = (w^1,\dots,w^n)$ and $\nu = (\nu^1,\dots,\nu^m)$ are independent Brownian notions defined on a probability space (Ω, F, P).

Furthermore, we assume

A_4: $h : R^d \to R^m$ is Borel measurable, continuously differentiable in x and

$$|h(t,x)| + |h_x(t,x)| \le K_3.$$

REMARK 2.1. These hypotheses can be weakened to those discussed by Haussman [4]. See [1].

Write $\hat{P}$ for the Wiener measure on $C([0,T],R^n)$ and μ for the Wiener measure on $C([0,T],R^m)$.

$$\Omega = C([0,T],R^n) \times C([0,T],R^m)$$

and the coordinate functions in Ω will be denoted (x_t, y_t). Wiener measure P on Ω is

$$P(dx, dy) = \hat{P}(dx)\mu(dy).$$

Definition 2.2. *$Y = \{Y_t\}$ will be the right continuous, complete filtration on $C([0,T], R^m)$ generated by*

$$Y_t^0 = \sigma\{y_s : s \le t\}.$$

The set of admissible control functions $\underline{U}$ will be the Y-predictable functions defined on $[0,T] \times C([0,T], R^m)$ with values in U.

For $u \in \underline{U}$ and $x \in R^d$, $\xi^u_{s,t}(x)$ will denote the strong solution of (2.1) corresponding to u with $\xi^u_{s,s} = x$.

Define

$$Z^u_{s,t}(x) = \exp\left(\int_s^t h(\xi^u_{s,r}(x))' dy_r - \frac{1}{2}\int_s^t h(\xi^u_{s,r}(x))^2 dr\right). \tag{2.3}$$

Note a version of Z defined for every trajectory y can be obtained by integrating the stochastic integral in the exponential by parts.

If a new probability measure P^u defined on Ω by putting

$$\frac{dP^u}{dP} = Z^u_{0,T}(x_0),$$

under P^u $(\xi^u_{0,t}(x_0), y_t)$ is a solution of the system (2.1) and (2.2). That is, under P^u, $\xi^u_{0,t}(x_0)$ remains a strong solution of (2.1) and there is an independent Brownian motion ν such that y_t satisfies (2.2).

Because of hypothesis A_4, for $0 \le t \le T$ easy applications of Burkholder's and Gronwall's inequalities show that

$$E[(Z^u_{0,t}(x_0))^p] < \infty \tag{2.4}$$

for all $u \in \underline{U}$ and all p, $1 \le p < \infty$.

COST 2.3. We shall suppose the cost is purely terminal and equals

$$c(\xi^u_{0,T}(x_0))$$

where c is a bounded, differentiable function. If control $u \in \underline{U}$ is used the expected cost is

$$J(u) = E_u[c(\xi^u_{0,T}(x_0))].$$

With respect to P, under which y_t is a Brownian motion

$$J(u) = E[Z^u_{0,T}(x_0)c(\xi^u_{0,T}(x_0))]. \tag{2.5}$$

A control $u^* \in \underline{U}$ is optimal if

$$J(u^*) \le J(u)$$

for all $u \in \underline{U}$. We shall suppose there is an optimal control u^*.

3. FLOWS.

For $u \in \underline{U}$ and $x \in R^d$ consider the strong solution

$$\xi^u_{s,t}(x) = x + \int_s^t f(r, \xi^u_{s,r}(x), u_r)dr + \int_s^t g(r, \xi^u_{s,r}(x))dw_r. \tag{3.1}$$

We wish to consider the behaviour of $\xi^u_{s,t}(x)$ for each trajectory y of the observation process. In fact the results of Bismut [3] and Kunita [6] extend and show the map

$$\xi^u_{s,t} : R^d \to R^d$$

is, almost surely, a diffeomorphism for each $y \in C([0,T], R^m)$.

Write

$$\|\xi^u(x_0)\|_t = \sup_{0 \le s \le t} |\xi^u_{0,s}(x_0)|.$$

Then, using Gronwall's and Jensen's inequalities, for any p, $1 \le p < \infty$

$$\|\xi^u(x_0)\|_T^p \le C\left(1 + |x_0|^p + \left|\int_0^T g(r, \xi^u_{0,r}(x_0))dw_r\right|^p\right)$$

almost surely, for some constant C.

Using A_3 and Burkholder's inequality

$$\|\xi^u(x_0)\|_T \in L^p \quad \text{for} \quad 1 \le p < \infty.$$

Suppose u^* is an optimal control, and write

$$\xi^*_{s,t}(\cdot) \quad \text{for} \quad \xi^{u^*}_{s,t}(\cdot).$$

The Jacobian $\frac{\partial \xi^*_{s,t}}{\partial x}$ is the matrix solution C_t of the equation

$$dC_t = f_x(t, \xi^*_{s,t}(x), u^*)C_t dt + \sum_{i=1}^n g_x^{(i)}(t, \xi^*_{s,t}(x))C_t dw_t^i. \tag{3.2}$$

with $C_s = I$.

Here $g^{(i)}$ is the i^{th} column of g and I is the $n \times n$ identity matrix. Writing $\|C\|_T = \sup_{0 \le s \le t} |C_s|$ and using Burkholder's, Jensen's and Gronwall's inequalities we see $\|C\|_T \in L^p$, $1 \le p < \infty$.

Consider the matrix valued process D defined by

$$D_t = I - \int_s^t D_r f_x(r, \xi^*_{s,r}(x), u^*_r)dr$$
$$-\sum_{i=1}^n \int_s^t D_r g_x^{(i)}(r, \xi^*_{s,r}(x))dw^i_r + \sum_{i=1}^n \int_s^t D_r (g_x^{(i)}(r, \xi^*_{s,r}(x)))^2 dr \qquad (3.3)$$

Then as in [5] or [6] $d(D_t C_t) = 0$ and $D_s C_s = I$ so

$$D_t = C_t^{-1} = \left(\frac{\partial \xi^*_{s,t}}{\partial x}\right)^{-1}.$$

Furthermore, $\|D\|_t \in L^p$, $1 \le p < \infty$.

Suppose $z_t = z_s + A_t + \sum_{i=1}^n \int_s^t H_i dw^i_r$ is a d-dimensional semimartingale. Bismut [3] shows one can consider the process $\xi^*_{s,t}(z_t)$ and in fact:

$$\xi^*_{s,t}(z_t) = z_s + \int_s^t \Big(f(r, \xi^*_{s,r}(z_r), u^*_r)$$
$$+ \sum_{i=1}^n g_x^{(i)}(r, \xi^*_{s,r}(z_r), u^*_r)\, \frac{\partial \xi^*_{s,r}}{\partial x} H_i$$
$$+ \frac{1}{2}\sum_{i=2}^n \frac{\partial^2 \xi^*_{s,r}}{\partial x^2}(H_i, H_i)\Big) dr$$
$$+ \int_s^t \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) dA_r + \sum_{i=1}^n \int_s^t \Big(g^{(i)}(r, \xi^*_{s,r}(z_r)) + \frac{\partial \xi^*_{s,r}}{\partial x}(z_r) H_i \Big) dw^i_r. \qquad (3.4)$$

Definition 3.1. *For $s \in [0,T]$, $h > 0$ such that $0 \le s < s+h \le T$, for any $\tilde{u} \in \underline{U}$, and $A \in Y_s$ consider a 'strong' variation u of u^* defined by*

$$u(t,w) = \begin{cases} u^*(t,w) & \text{if } (t,w) \notin [s, s+h] \times A \\ \tilde{u}(t,w) & \text{if } (t,w) \in [s, s+h] \times A. \end{cases}$$

Theorem 3.2. *For any strong variation u of u^* consider the process*

$$z_t = x + \int_s^t \left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))dr. \qquad (3.5)$$

Then the process $\xi^*_{s,t}(z_t)$ is indistinguishable from $\xi^u_{s,t}(x)$.

PROOF. We shall substitute in (3.4), (noting $H_i = 0$ for all i). Therefore,

$$\xi^*_{s,t}(z_t) = x + \int_s^t f(r, \xi^*_{s,r}(z_r), u^*_r)dr$$
$$+ \int_s^t \left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)\left(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\right)^{-1} (f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(z_r), u^*_r))dr$$
$$+ \int_s^t g(r, \xi^*_{s,r}(z_r))dw_r.$$

The solution of (3.1) is unique, so $\xi^*_{s,t}(z_t) = \xi^u_{s,t}(x)$. Note $u(t) = u^*(t)$ if $t > s+h$ so $z_t = z_{s+h}$ if $t > s+h$ and

$$\xi^*_{s,t}(z_t) = \xi^*_{s,t}(z_{s+h})$$
$$= \xi^*_{s+h,t}(\xi^u_{s,s+h}(x)). \qquad (3.6)$$

4. THE EXPONENTIAL DENSITY.

Consider the $(d+1)$-dimensional system

$$\xi^*_{s,t}(x) = x + \int_s^t f(r, \xi^*_{s,r}(x), u^*_r)dr + \int_s^t g(r, \xi^*_{s,r}(x))dw_r$$
$$Z^*_{s,t}(x,z) = z + \int_s^t Z^*_{s,r}(x,z) h(\xi^*_{s,r}(x))' dy_r. \qquad (4.1)$$

That is, we are considering an augmented flow (ξ, Z) in R^{d+1} in which Z^* has a variable initial condition $z \in R$. Note:

$$Z^*_{s,t}(x,z) = z Z^*_{s,t}(x).$$

The map $(x,z) \to (\xi^*_{s,t}(x), Z^*_{s,t}(x,z))$ is, almost surely, a diffeomorphism of R^{d+1}. Clearly,

$$\frac{\partial \xi^*_{s,t}}{\partial z} = 0, \qquad \frac{\partial f}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial g}{\partial z} = 0.$$

The Jacobian of this augmented map is represented by the matrix

$$\tilde{C}_t = \begin{pmatrix} \frac{\partial \xi^*_{s,t}}{\partial x} & 0 \\ \frac{\partial Z^*_{s,t}}{\partial x} & \frac{\partial Z^*_{s,t}}{\partial z} \end{pmatrix}.$$

In particular, from (4.1), for $1 \le i \le d$

$$\frac{\partial Z^*_{s,t}}{\partial x_i} = \sum_{j=1}^m \int_s^t \Big(Z^*_{s,r}(x,z) \sum_{k=1}^n \frac{\partial h^j}{\partial \xi_k} \cdot \frac{\partial \xi^*_{k,s,r}}{\partial x_i} + h^j\left(\xi^*_{s,r}(x)\right) \frac{\partial Z^*_{s,r}}{\partial x_i}\Big) dy^j_r. \qquad (4.2)$$

We are interested in solutions of (4.1) and (4.2) only when $z = 1$, so as above we write

$$Z^*_{s,t}(x) \quad \text{for} \quad Z^*_{s,t}(x,1) \quad \text{etc.}$$

Lemma 4.1.

$$\frac{\partial Z^*_{s,t}}{\partial x} = Z^*_{s,t}(x)\Big(\int_s^t h_x(\xi^*_{s,t}(x)) \cdot \frac{\partial \xi^*_{s,r}}{\partial x} d\nu_r\Big)$$

where, as in (2.2), $d\nu_t = dy_t - h(\xi^*_{s,t}(x))dt$.

PROOF. From (4.2)

$$\frac{\partial Z^*_{s,t}}{\partial x} = \int_s^t \Big(\frac{\partial Z^*_{s,r}}{\partial x} h'(\xi^*_{s,r}(x)) + Z^*_{s,r}(x) h_x(\xi^*_{s,r}(x)) \frac{\partial \xi^*_{s,r}}{\partial x}\Big) dy_r. \tag{4.3}$$

Write

$$L_{s,t}(x) = Z^*_{s,t}(x)\Big(\int_s^t h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} d\nu_r\Big).$$

Then

$$Z^*_{s,t}(x) = 1 + \int_s^t Z^*_{s,r}(x) h'(\xi^*_{s,r}(x)) dy_r$$

and the product rule gives

$$L_{s,t}(x) = \int_s^t L_{s,r}(x) h'(\xi^*_{s,r}(x)) dy_r + \int_s^t Z^*_{s,r}(x) h_x \cdot \frac{\partial \xi^*_{s,r}}{\partial x} dy_r.$$

Consequently, $L_{s,t}(x)$ is also a solution of (4.3), so by uniqueness

$$L_{s,t}(x) = \frac{\partial Z^*_{s,t}}{\partial x}.$$

LEMMA 4.2. *If z_t is as defined in (3.5)*

$$Z^*_{s,t}(z_t) = Z^u_{s,t}(x).$$

PROOF.

$$Z^u_{s,t}(x) = 1 + \int_s^t Z^u_{s,r}(x) h'(\xi^u_{s,r}(x)) dy_r. \tag{4.4}$$

Applying (3.4) to $Z^*_{s,t}(z_t)$ we see:

$$Z^*_{s,t}(z_r) = 1 + \int_s^t Z^*_{s,r}(z_r) h'(\xi^*_{s,r}(z_r)) dy_r$$

$$= 1 + \int_s^t Z^*_{s,r}(z_r) h'(\xi^u_{s,r}(x)) dy_r$$

by Theorem 3.2. However, (4.4) has a unique solution so

$$Z^*_{s,t}(z_r) = Z^u_{s,t}(x).$$

Again, note that for $t > s + h$

$$Z^*_{s,t}(z_t) = Z^*_{s,t}(z_{s+h}). \tag{4.5}$$

5. THE ADJOINT PROCESS.

u^* will be an optimal control and u a perturbation of u^* as in Definition 3.1. Again write

$$x = \xi^*_{0,s}(x_0).$$

The minimum cost is

$$J(u^*) = E[Z^*_{0,T}(x_0) c(\xi^*_{0,T}(x_0))]$$
$$= E[Z^*_{0,s}(x_0) Z^*_{s,T}(x) c(\xi^*_{s,T}(x))].$$

Also,

$$J(u) = E[Z^*_{0,s}(x_0) Z^u_{s,T}(x) c(\xi^u_{s,T}(x))]$$
$$= E[Z^*_{0,s}(x_0) Z^*_{s,T}(z_{s+h}) c(\xi^*_{s,T}(z_{s+h}))]$$

by (3.6) and (4.5). Recall $Z^*_{s,T}(\cdot)$ and $c(\xi^*_{s,T}(\cdot))$ are differentiable almost surely, with continuous and uniformly integrable derivatives. Consequently, writing

$$\Gamma(s, z_r) = Z^*_{0,s}(x_0) Z^*_{s,T}(z_r)\Big\{c_\xi(\xi^*_{s,T}(z_r)) \frac{\partial \xi^*_{s,T}}{\partial x}(z_r) + c(\xi^*_{s,T}(z_r))\Big(\int_s^T h_\xi(\xi^*_{s,\sigma}(z_r)) \frac{\partial \xi^*_{s,\sigma}}{\partial x}(z_r) d\nu_\sigma\Big)\Big\}\Big(\frac{\partial \xi^*_{s,r}}{\partial x}(z_r)\Big)^{-1}$$

for $s \le r \le s + h$, we have

$$J(u) - J(u^*) = E[Z^*_{0,s}(x_0)\{Z^*_{s,t}(z_{s+h}) c(\xi^*_{s,t}(z_{s+h})) - Z^*_{s,T}(x) c(\xi^*_{s,T}(x))\}]$$
$$= E\Big[\int_s^{s+h} \Gamma(s, z_r)(f(r, \xi^*_{s,r}(z_r), u_r) - f(r, \xi^*_{s,r}(x), u^*_r)) dr\Big]. \tag{5.1}$$

This formula describes the change in the expected cost arising from the perturbation u of the optimal control. However, $J(u) \ge J(u^*)$ for all $u \in \underline{U}$ so the right hand side of (5.1) is non-negative for all $h > 0$. We wish to divide by $h > 0$ and let $h \to 0$. This requires some careful arguments using the uniform boundedness of the random variables and the monotone class theorem. It can be shown that there is a set $S \subset [0, T]$ of zero Lebesgue measure such that if $s \notin S$

$$E[\Gamma(s, x)(f(s, \xi^*_{0,s}(x_0), u) - f(s, \xi^*_{0,s}(x_0), u^*_s)) I_A] \ge 0 \tag{5.2}$$

for any $u \in U$ and $A \in Y_s$.

Details of this argument can be found in [1]. Define

$$p_s(x) = E^*\Big[c_\xi(\xi^*_{0,T}(x_0)) \frac{\partial \xi^*_{s,T}}{\partial x}(x) + c(\xi^*_{0,T}(x_0))\Big(\int_s^T h_\xi(\xi^*_{0,\sigma}(x_0)) \frac{\partial \xi^*_{s,\sigma}}{\partial x}(x) d\nu_\sigma\Big) \Big| Y_s \vee \{x\}\Big]$$

where $x = \xi^*_{0,s}(x_0)$ and E^* is the expectation under $P^* = P^{u^*}$.

In (5.2) we have established the following:

Theorem 5.1. *$p_s(x)$ is the adjoint process for the partially observed optimal control problem. That is, if $u^* \in \underline{U}$ is optimal there is a set $S \subset [0,T]$ of zero Lebesgue measure such that for $s \notin S$*

$$E^*[p_s(x)f(s,x,u^*) \mid Y_s] \geq E^*[p_s(x)f(s,x,u) \mid Y_s] \quad \text{a.s.} \tag{5.3}$$

so the optimal control u^* almost surely minimizes the conditional Hamiltonian.

If $x = \xi^*_{0,s}(x_0)$ has a conditional density $q_s(x)$ under P^*, and if f is differentiable in u, (5.3) implies

$$\sum_{i=1}^{k}(u_i(s) - u_i^*(s)) \int_{R^d} \Gamma(s,x)\, \frac{\partial f}{\partial u_i}\,(s,x,u^*)q_s(x)dx \geq 0.$$

This is the result of Bensoussan [2].

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