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Decentralized Stabilization of Large Scale Systems by Frequency Domain Methods

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Abstract

A notion of weak coupling between subsystems of a large-scale system is developed based on a sufficient condition for a linear, time-invariant MIMO systems. The weak coupling measure provides a frequency dependent requirement on the system response for stability of a decentralized controller for the system. An abstract notion of a generalized Nyquist criterion for MIMO systems (as discussed in Brockett and Byrnes [7]) is extended for decentralized feedback control. Computational methods are provided which exploit a connection between a gap-metric and the principal angles between pairs of subspaces. The principal angles are shown to be related to the singular values of an easily computed matrix.

1.0 Introduction

The area of control of large scale systems deals with control problems for large dimensional multivariable plants. Typically several models for the plant dynamics may be developed based on various levels of detailed subsystem models and their interconnections at a microscopic level to various level of aggregate or macroscopic models of the overall plant. These various modeling levels have been used to advantage in the design methods for large scale problems and are reflected in various decentralized and hierarchical control structures.

As a first step in many such procedures a simple decentralized feedback controller is sought where the decentralization is consistent in some method specific technical sense with the model structure. The objective of the decentralized feedback controller typically comprises some low level compensation providing simplified interface to hierarchical controller. The success of such a decentralized compensation scheme is often based on an underlying model structure assumption. One such model structure assumption which has been exploited in the design and analysis of decentralized feedback is the notion of weakly interacting subsystems.

Although various researchers have provided conditions for testing the weak interacting hypothesis on a system model (cf. Aplevich [6]) we will be more concerned with conditions providing for the stability of decentralized feedback. Results along these lines have been obtained by Cook [2], Araki [1], Lasley and Michel [3] for a fairly general class of systems including linear, time-varying, and including certain types of memoryless nonlinearities. These approaches employ various function space norms representing the system impulse response or transfer function and may lead to rather gross estimates of the robustness of a decentralized feedback.

In some earlier work the present authors [5] have explored the use of a frequency domain notion of weakly-interacting systems towards the goal of establishing sharper estimates of stability margins for the decentralized feedback control system. The notion of weak coupling described in [5] is based on the idea of a block diagonally dominant (BDD) transfer function matrix. The BDD concept described in [5] is motivated as a generalization of the Inverse Nyquist Array (INA) method of Rosenbrock [4]. However it is clear that the design method of [5] differs from the design methods of Rosenbrock and his colleagues in that:

1) whereas in INA methods one may be required to provide series multivarible compensation to achieve open-loop diagonal dominance, in our method no attempt is made to modify the natural open-loop subsystem coupling

2) whereas in INA methods the MIMO design problem is reduced to a series of SISO design problems for which classical design methods can be applied, in our methods the large scale MIMO problem is reduced to a series of smaller MIMO problems

3) whereas in INA methods the frequency dependent Gershgorin sets for the transfer function matrix sweep out a fuzzy Nyquist locus for each of the SISO feedback designs, in our method the block generalized Gershgorin sets provide <u>no</u> such interpretation.

Technical details of remark 3 above are discussed at length in [5]. However, we can say loosely that the problem - up until now - has been that one was not sure what is the appropriate notion of a generalized Nyquist locus for the local MIMO design problems which can capture the effect all of the local MIMO compensators in the overall decentralized design. This is the contribution of the current paper.

The format of the paper is as follows. In section 2 we briefly review several notions of generalized Nyquist criterion for MIMO feedback focusing on the version we propose to employ for which the Nyquist locus becomes a curve on a complex Grassman manifold. In section 3 we review several metric functions which can provide a basis for a topology of the Grassman manifold. Focusing on a notion of "near intersections" between subspaces of a unitary space we discuss the relationship between principal angles between subspaces and the singular values of a certain matrix [13]. Finally in section 4 we provide several results addressing the main problem of decentralized feedback stabilization and provide a new notion of weakly coupled systems for which broad or fuzzy Nyquist loci are established for the individual local MIMO feedback designs comprising the decentralized feedback control system.

Throughout the paper we will use the following notation. Capital letters will be matrices and if they are underscored then they represent finite dimensional subspaces according to context.

2.0 A Generalized Nyquist Criterion for MIMO Systems

The use of Nyquist type arguments in providing tests for the stability of feedback systems has been implicit in the development of frequency domain methods for MIMO systems over the past decade. In the early 1970's Rosenbrock [4] applied the principle of the argument to the rational function $\det[I + FG(s)]$ associated with the feedback equations

$$y(s) = G(s)\, u(s)$$
$$u(s) = Fy(s) + r(s) \qquad (2.1)$$

where $y(s)$ is p-vector of outputs and $u(s)$ is m-vector of inputs. More recently Postlethwaite and MacFarlane [9] and Desoer and Wang [10] have developed explicitly a generalized Nyquist criterion for the special case of $F=fI_p$ where $p=m$. Using this special structure Postethwaite and MacFarlane [9] are able to develop bounds for a notion of gain and phase margin for MIMO feedback. In these cases the generalized Nyquist locus is taken as the p eigenloci of the return difference matrix $I + FG(s)$ evaluated along the imaginary axis in the complex plane. Doyle and Stein [11] point out that since the eigenvalues of the return difference may be very sensitive to general perturbations in the matrix $G(j\omega)$ one should look at the frequency dependent singular values of the return difference in assessing robustness.

A geometric viewpoint is taken in Brockett and Byrnes [7] in describing a generalized Nyquist criterion. Here for the first time the general case of F mxp with $p\neq m$ is treated, although somewhat abstractly. Significantly, their approach avoids formulation of the return difference matrix and as a result allows the separate characterization of an abstract critical point (resulting from F) and a Nyquist locus (resulting from $G(j\omega)$). This formulation preserves most nearly the practical aspects of Nyquist criterion exploited in SISO system design [4].

The setup is as follows. The feedback equations 2.1 are written in matrix form as

$$\begin{bmatrix} G(s) & -I_p \\ I_m & F \end{bmatrix} \begin{pmatrix} u(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (2.1')$$

where it is clear that a complex scalar s is a closed loop pole if and only if the $\ker[G(s), -I_p]$ intersects $\ker[I_m, F]$ in some nontrivial way. In the case that $p\leq m$ we can construct an abstract Nyquist locus Γ_G for the pxm transfer function matrix $G(s)$ by thinking of Γ_G as the image of the imaginary axis under the map $\underline{G}(s) = \ker[G(s), -I_p]$. For each $s=j\omega$ the object $\underline{G}(s) = \ker[G(s), -I_p]$ is a p-dimensional subspace of a p+m dimensional complex space. Thus Γ_G can be thought of as a "curve" in the complex Grassmanian $\Gamma_G \subseteq \text{Grass}(p,m+p)$ of p-dimensional planes in m+p space. The question of whether $\underline{G}(s) = \ker[G(s), -I_p]$ intersects $\underline{F} = \ker[I_m, F]$ can be ascertained by utilizing the dual structure of the Grassmanian in the following way. The Schubert hypersurface associated with an m-dimensional subspace of C^{m+p} is defined as

$$\sigma(\underline{F}) = \{\underline{M} \in \text{Grass}(p, m+p): \dim(\underline{M}\cap\underline{F}) > 0\},$$

a hypersurface in Grass(p, m+p) representing the point $\underline{F}\in\text{Grass}(m, m+p)$, the dual of Grass(p, m+p). Then $\underline{F}$ intersects $\underline{G}(s)$ if $\underline{G}(s)$ intersects $\sigma(\underline{F})$ in Grass(p,m+p).

Using this dual structure (cf. [7] & [8] for details) the following theorem is provided.

Theorem 2.1: (Generalized Nyquist Theorem):

Suppose $G(s)$ is a proper rational pxm

transfer function matrix with no poles on Res=0. Suppose the abstract Nyquist locus Γ_G does not intersect the Schubert hypersurface $\sigma(\underline{F})$ defined by the feedback matrix F. Let p_o be the number of open loop poles of G(s) in the closed right half plane (CRHP) and p_c be the number of closed loop poles in CRHP. Then

$$p_c = p_o + \rho$$

where ρ is the number of encirclements of the abstract Nyquist locus Γ_G about the Schubert hypersurface $\sigma(\underline{F})$ taken in a positive direction.

Proof: (cf. [7])

Clearly theorem 2.1 does not admit any readily obvious graphical representation that would permit the determination of the winding number ρ (except in trivial cases). However the theorem does permit us to ascertain the stability of a MIMO feedback system involving a plant $G_1(s)$ with feedback F_1 by testing for homotopical equivalence with some other feedback system $G_2(s)$ with F_2 (of appropriate dimensions) which is known to be stable. To show such equivalence we will need a measure of how close a point $\underline{G}(s) \in$ Grass(p, m+p) is to some Schubert hypersurface $\sigma(\underline{F})$ Grass(p, m+p).

3.0 Topology of "Near" Intersecting Subspaces: Equivalence of some well known metrics on the Grassman Space

3.1 Plücker Metric

As discussed above the Grassman space, Grass(p,n), is the space of all p dimensional subspaces of C^n. Clearly for any $\underline{N} \in$ Grass(p,n) we can express p basis vectors for $\underline{N}$ in terms of some coordinate systems by writing an nxp matrix say B_1. In terms of some other coordinate system we can write a new matrix B_2. In this case there exists a nonsingular pxp matrix A such that $B_1 = B_2 A$ and conversely. Thus $\underline{N}$=image(B_1)=image(B_2).

Following Byrnes, et al [8] we state

Definition 3.1: The Plücker coordinates of a pxn matrix B is the $\binom{p}{n}$ dimensional vector of determinants of all pxp submatrices of B.

It is easy to show that for any B_1 and B_2 as above their respective Plücker coordinates will differ by a scalar multiple of each other; i.e., their Plücker coordinates are aligned. One way to provide a notion of distance on Grass(p,n) is then to think of any two subspaces of dimension p say $\underline{N},\underline{M} \in$ Grass(p,n) in terms of their Plücker coordinates (say $u_N, v_M, \in C\binom{p}{n}$). Then the function

$$d(\underline{N},\underline{M}) = (1 - u_N^* v_M)/||u_N||_2 \cdot ||v_M||_2 \qquad (3.1)$$

is a metric and obeys the property

$$-1 \leq d(\underline{N},\underline{M}) \leq 1.$$

3.2 Gap-Metric

Another sort of metric can be described by thinking of points in Grass(p,n) in terms of the p-dimensional subspaces of C^n that they represent. Here we take an abstract "basis-free" viewpoint in describing the subspaces.

Let $\underline{M}$, $\underline{N} \subsetneq C^n$ be subspaces. Let dim $\underline{M}$ = p and dim $\underline{N}$ = m. The following function called the gap (or aperture) between $\underline{M}$ and $\underline{N}$ is defined in Kato [11].

Definition 3.2: The gap between $\underline{M}$ and $\underline{N}$ is

$$\delta(M,N) = \left\{ \max \sup_{\substack{||x||=1 \\ x\in\underline{M}}} \inf_{y\in\underline{N}} ||x-y||, \sup_{\substack{||y||=1 \\ y\in\underline{N}}} \inf_{x\in\underline{M}} ||y-x|| \right\}. \qquad (3.2)$$

The gap function obeys the following properties which follow immediately from the definition.

(P1) $\delta(\underline{M},\underline{N}) = 0$ if and only if $\underline{M} = \underline{N}$
(P2) $\delta(\underline{M},\underline{N}) = \delta(\underline{N},\underline{M})$
(P3) $0 \leq \delta(\underline{M},\underline{N}) \leq 1$

Furthermore the gap function obeys the following property which will be significant for us:

(P4) $\delta(\underline{M},\underline{N}) < 1$ if and only if dim$(\underline{M})$ = dim$(\underline{N})$

(cf. [1, pg. 200]). In general the gap $\delta(\underline{M},\underline{N})$ is not a metric. However modifying the definition slightly by taking infimums over the appropriate unit ball in each subspace the resulting modified gap $\hat{\delta}(\underline{M},\underline{N})$ obeys the triangle inequality and thus becomes a metric.

Property (P4) clearly indicates that using this modified "gap-metric" one can actually form a basis for a topology of the Grassman space via neighborhoods of the form

$$B_\epsilon(M) = \{\underline{N} \in \text{Grass}(p,n): \hat{\delta}(\underline{M},\underline{N}) < \epsilon < 1\} .$$

3.3 Orthogonal Projections in Unitary Space and the Gap Metric

In a unitary space E^n we can employ the notion of an orthogonal projector to represent a subspace $\underline{E} \subseteq E^n$; e.g., take $E^n = C^n$ and the natural inner product $\langle x,y \rangle = \bar{x}^T y$. If P_M(resp. P_N) is an orthogonal projector whose range is the subspace $\underline{M} \subseteq E^n$ (resp. $\underline{N} \subseteq E^n$) then using the natural Euclidean norm we can state the following

Theorem 3.3:

$$\delta(\underline{M},\underline{N}) = ||P_M - P_N||_2 . \qquad (3.3)$$

Proof: (cf. Kato [11]).

Property (P4) of the gap is then related to the following fact.

Theorem 3.4:

Any two orthogonal projectors P_M, P_N which satisfy $||P_M - P_N|| < 1$ are unitarily equivalent. That is there exists a unitary transformation U such that $UP_MU^* = P_N$. (U is unitary if $U^*U = I$).

Proof: (cf. Kato [11]).

Unitary transformations have an intuitive geometric appeal because they represent orthogonal rotations of the given vector space coordinate system. Thus we see that with the structure of a unitary space the gap-metric (here the gap function $\delta(\cdot,\cdot)$ becomes naturally a metric) takes on a particularly natural geometric appeal. Indeed, in finite dimensional unitary spaces, for which we have interest, the transformation U of theorem 3.4 can be represented by an easily computable matrix. In Kato [11] these results (and others) are used to study perturbations of linear operations on infinite dimensional spaces. In Stewart [2] similar ideas are applied to certain numerical problems in the computation of invariant subspaces for matrix (finite dimensional) operators. As we discuss in the subsequent sections our concern is slightly different but will follow along the same line of reasoning.

3.4 Intersection Between Subspaces and the Dual Structure of the Grassman Space

From the statement of the generalized Nyquist criterion above it is clear that we will be interested in characterizing the "near" intersection between certain pairs of subspaces. On the Grassmanian manifold this is characterized by an intersection between an element $\underline{M} \in Grass(p,n)$ and a Schubert hypersurface $\sigma(\underline{N})$ $\overline{Grass}(p,n)$ associated with the subspace $\underline{N} \in Grass(n-p,n)$.

Towards this end we provide the following:

Definition 3.4:

Let $\underline{M} \subseteq C^n$ be a p-dimensional subspace and $\underline{N} \subseteq C^n$ an m-dimensional subspace. Then the minimum gap between $\underline{M}$ and $\underline{N}$ in C^n is

$$\gamma(\underline{M},\underline{N}) = \min \left\{ \inf_{\substack{x \in \underline{M} \\ ||x||=1}} \inf_{y \in \underline{N}} ||x-y||, \inf_{\substack{y \in \underline{N} \\ ||y||=1}} \inf_{x \in \underline{M}} ||y-x|| \right\}. \qquad (3.4)$$

Obviously, the minimum gap satisfies the properties:

(P1) $0 \leq \gamma(\underline{M},\underline{N}) \leq \delta(\underline{M},\underline{N})$

(P2) $\gamma(\underline{M},\underline{N}) = 0$ if and only if $\dim(\underline{M} \cap \underline{N}) > 0$.

Based on (P2) it is clear that for the abstract Nyquist criterion described in section 2.0 that the min-gap can provide a measure of distance between the abstract Nyquist contour Γ_G and the abstract critical point $\sigma(\underline{F})$ as

$$\min_{s \text{ on } D} \gamma(\underline{G}(s),\underline{F}) \qquad (3.5)$$

where $\underline{G}(s) \in Grass(m,n)$ and $\underline{F} \in Grass(p,n)$. This can be considered a geometric construction of a "stability margin" for MIMO feedback.

Following the line of reasoning of section 3.3 we make the following claim.

Corollary 3.5:

If P_M and P_N are both orthogonal projectors in C^n with image $(P_M) = \underline{M}$, image $(P_N) = \underline{N}$ then

$$\gamma(\underline{M},\underline{N}) = ||(P_M - P_N)^{-1}||^{-1} . \qquad (3.6)$$

3.5 Canonical Angles Between Subspaces

There is a natural notion of angles between pairs of subspaces in a unitary space. In finite dimensional spaces these angles can be computed from singular values of a particular matrix. If we let $\underline{M},\underline{N}$ be a pair of subspaces of C^n with dim $\underline{M}$=p dim $\underline{N}$=m. Assume m≥p. Then we say the smallest angles between $\underline{M}$ and $\underline{N}$ (cf. [13]) $\theta_1(\underline{M},\underline{N}) = \theta_1 \in [0,\pi/2]$ is given by

$$\cos\theta_1 = \max_{\substack{u \in M \\ ||u||_2=1}} \max_{\substack{v \in N \\ ||v||_2=1}} u^* v . \qquad (3.7)$$

Following Björck & Golub [13] we define recursively the principal angles θ_k k=1,...,p as follows

Definition 3.6:

The principal angles $\theta_k \in [0 \ \pi/2]$ between $\underline{M}$

and $\underline{N}$ are given recursively for $k=1,2,\ldots,p$ by

$$\cos \theta_k = \max_{\substack{u \in \underline{M} \\ \|u\|_2 = 1}} \ \max_{\substack{v \in \underline{N} \\ \|v\|_2 = 1}} \ u^* v = v_k{}^* v_k \tag{3.8}$$

subject to the constraints

$$u_j^* u = 0 \text{ and } v_j^* v = 0 \text{ for } j=1,\ldots,k-1. \tag{3.9}$$

We call the vectors $(u_1,\ldots,u_p)$ and $(v_1,\ldots v_p)$ the <u>principal vectors</u> for the pair of subspaces.

In this section we review how the principal angles can be computed for a pair of subspaces. The relation between certain principal angles and the gap will be clarified using orthogonal projectors. The result will be a computational procedure for determining the gap $\delta(.,.)$ and the min-gap $\gamma(.,.)$ between a pair of subspaces $\underline{M}$, $\underline{N}$. Moreover using the principal vectors we can compute a basis for the intersection $\underline{M} \cap \underline{N}$. For the problem of multivariable feedback such a basis can be used to describe how certain modal behavior of the system is reflected from an input-output view point. In section 4.0 we provide (based on these ideas) a new notion of weakly interacting subsystems with respect to stability using decentralized feedback.

The main computational result which we exploit requires that we have a unitary basis for each of the subspaces $\underline{M}$ and $\underline{N}$. Since this can be obtained conceptually using a Gram-Schmidt procedure (and in practice using Householder reflections) we assume that we have a pair of matrices $Q_M \in C^{n \times p}$ with $Q_N^* Q_N = I_m$ and $Q_M^* Q_M = I_p$.

<u>Theorem 3.7</u>:

Given Q_M and Q_N such that image $(Q_M) = \underline{M}$ and image $(Q_N) = \underline{N}$ each a subspace of C^n. Compute the singular value decomposition (SVD) of

$$Q_M^* \, Q_N = Y_M \, C \, Y_N^* \tag{3.10}$$

where

$$Y_M^* \, Y_M = Y_M \, Y_M^* = Y_N^* \, Y_N = I_p \tag{3.11}$$

and $C = \cos\theta = \text{diag}(\sigma_1,\ldots,\sigma_p)$

$\theta = \text{diag}(\theta_1,\ldots,\theta_p)$

and $\sigma_1 \geq \ldots \geq \sigma_p$. Then $\theta_1 \leq \ldots \leq \theta_p$ are the principal angles between $\underline{M}$ and $\underline{N}$ the columns of and $U = Q_M Y_M$, $V = Q_N Y_N$ are the principal vectors.

<u>Proof</u>: (cf. [13]).

<u>Corollary 3.8</u>:

Let $P_M = Q_M Q_M^*$ be an orthogonal projector on $\underline{M}$. Then compute the SVD of

$$(I - P_M)Q_N = W_M \, S \, Y_N^* \tag{3.12}$$

where $S = \sin\theta$. Here W_M gives the principal vectors in the complement $\underline{M} = \text{image}(Q_M)$ associated with the pair of subspaces $\underline{M}$, $\underline{N}$.

<u>Theorem 3.9</u>:

As above let P_M and P_N be orthogonal projectors on M and N respectively. Then the nonzero eigenvalues of $P_M - P_N$ are $\pm \sin \theta_i$ for $i=1,\ldots,p$.

Finally we can state as a corollary to theorem 3.9

<u>Corollary 3.10</u>:

With the above notation

$$\delta(\underline{M},\underline{N}) = \|P_M - P_N\|_2 = \sin \theta_p \tag{3.13}$$

$$\gamma(\underline{M},\underline{N}) = \|(P_M - P_N)^{-1}\|_2^{-1} = \sin \theta_1 \tag{3.14}$$

<u>Proof</u>: See Theorem 3.2 and corollary 3.5.

4.0 Stability of Decentralized Feedback

The case of decentralized feedback can be expressed in terms of the feedback equations (2.1) as the case for which F has some known sparsity associated with the allowed information pattern for feedback. For example we consider the usual case where

$$F = \text{block diag}\{F_1,\ldots,F_k\} \tag{4.1}$$

with respect to some partitioning of F. The partition of F is induced by a partitioning of the space of inputs and outputs. Conformally a partition of G(s) is obtained. Clearly if the resulting G(s) is also block diagonal then the decentralized control problem is completely decoupled and we can proceed to design the F_i for $i=1,\ldots k$ local feedback compensators separately without further consideration for interactions.

Let $G^D(s)$ be the pxm rational transfer function matrix formed from G(s) as

$$[G^D(s)]_{ij} = \begin{cases} [G(s)]_{ii} \\ 0, \text{ for } i \neq j \end{cases} \tag{4.2}$$

for $i,j = 1,\ldots,k$ where $[G(s)]_{ij}$ is the ixj block submatrix (of dimension $p_i \times m_i$) of the partitioned form of $G(s)$. We next show that $G(s)$ can be considered weakly coupled with respect to the partitioning of F if $G^D(s)$ and $G(s)$ are close in technical sense to be made specific in the sequel.

From the point of view of stability it is natural to consider $\underline{G}(s) = \ker[G(s), -I_p]$ and $\underline{G}^D(s) = \ker[G^D(s), -I_p]$ as "close" if the orthogonal projectors P_{G^D} and P_G onto $\underline{G}^D(s)$ and $\underline{G}(s)$ respectively satisfy

$$||P_{G^D(s)} - P_{G(s)}||_2 = \delta\{\underline{G}^D(s), \underline{G}(s)\} < \epsilon(s) \quad (4.3)$$

for $\mathrm{Re}\, s = 0$. How small must $\epsilon(s)$ be for $\underline{F}$ = block diag$\{F_1 \ldots, F_k\}$ to be a stabilizing compensator will be established by the main result which follows.

Let D be the usual closed contour constructed of a relatively large portion of the $j\omega$ axis and a semicircular segment in the closed right half-plane (CRHP).

Theorem 4.1 (Main Result):

Let Γ_G be defined as the image of the closed contour D under the map $\ker[G(s), -I_p]$. With respect to the induced partition of $G(s)$ let Γ_i be defined similarly as

$$\underline{G}_i(s) = \ker[G_{ii}(s), -I_{pi}]: \quad D \to \Gamma_i.$$

As in theorem 2.1 let ρ be the number of encirclements of Γ_G (a curve in Grass(p,m+p)) about the Schubert hypersurface $\sigma(\underline{F})$ where $\underline{F} = \ker[I_m, F]$ and F being a block diagonal (decentralized) feedback compensator. Similarly let ρ_i be the winding number of Γ_i (a curve in Grass(p_i, m_i+p_i)) associated with $\sigma(\underline{F}_i)$ where $\underline{F}_i = \ker[I_{mi}, F_i]$ for each $i = 1,\ldots,k$. Then if

$$\min_{i=1,\ldots,k} \gamma(\underline{G}_i(s), \underline{F}_i) > \delta(\underline{G}(s), \underline{G}^D(s)) \quad (4.4)$$

for all s on D then

$$\rho = \sum_{i=1}^{k} \rho_i. \quad (4.5)$$

Proof:

First consider the following lemma.

Lemma:

Let $\underline{N}_1, \underline{N}_2 \subseteq C^n$ subspaces of dimension p, $\underline{N}_1, \underline{N}_2 \in$ Grass(p,n), and $\underline{N}_3 \in$ Grass(n-p, n). If

$$\gamma(\underline{N}_1, \underline{N}_3) > \delta(\underline{N}_1, \underline{N}_2) \quad (4.6)$$

then

$$\underline{N}_2 \cap \underline{N}_3 = \{0\}.$$

Proof of Lemma:

From (3.5) $\gamma(\underline{N}_1, \underline{N}_3)$ gives a measure of the distance (in the metric $\delta(\cdot,\cdot)$) between $\underline{N}_1 \in$ Grass(p,n) and the Schubert hypersurface $\sigma(\underline{N}_3) \subseteq$ Grass(p,n). The result of the lemma is then obvious by the δ-metric topology of Grass(p,n).

End proof of lemma . . .

Next define the matrix

$$G_\epsilon(s) = G^D(s) + \epsilon[G(s) - G^D(s)] \quad (4.7)$$

for $0 \le \epsilon \le 1$. The associated subspaces

$$\underline{G}_\epsilon(s) = \ker[G_\epsilon(s), -I_p]$$

will satisfy

$$\delta(\underline{G}(s), \underline{G}^D(s)) \ge \delta(\underline{G}_\epsilon(s), \underline{G}^D(s)) \quad (4.8)$$

for $0 \le \epsilon \le 1$ by the δ-metric topology of Grass(p, m+p). Also note that

$$\min_{i=1,\ldots,k} \gamma(\underline{G}_i(s), \underline{F}_i) = \gamma(\underline{G}^D(s), \underline{F})$$

where $\underline{G}^D(s) = \ker[G^D(s), -I_p]$. Thus clearly (4.4) guarantees under the assumptions given that if $\underline{G}^D(s)$ $\underline{F} = \{0\}$ for s on D then $\underline{G}(s)$ $\underline{F} = \{0\}$ for all s on D.

Let

$$A(\epsilon,s) = \begin{bmatrix} G_\epsilon(s) & -I_p \\ I_m & F \end{bmatrix}$$

and

$$A_i(s) = \begin{bmatrix} G_{ii}(s) & -I_{pi} \\ I_{mi} & F_i \end{bmatrix}.$$

Then let

$$\beta(\epsilon,s) = \frac{\det A(\epsilon,s)}{\prod_{i=1}^{k} \det A_i(s)}$$

map D into Γ_β a closed curve in the complex plane. Now Γ_β does not encircle the origin since otherwise there must exist some s on D and $0 \le \epsilon \le 1$ with $\beta(\epsilon,s) = 0$. However this means that $\underline{G}_\epsilon(s)$ intersects $\underline{F}$ in some nontrivial way -- a situation which is precluded by (4.4) using the lemma and (4.8). Finally application of the principle of the argument to $\beta(\epsilon,s)$ gives the result.

End proof of theorem . . .

Theorem 4.1 provides an appropriate generalization of the fuzzy Nyquist loci employed by Rosenbrock in Inverse Nyquist Array design. Associated with the k separate generalized Nyquist loci Γ_i $Grass(p_i, m_i+p_i)$ where $\underline{G}_i(s) : D \rightarrow \Gamma_i$ attach for each point on Γ_i a δ-neighborhood of radius $\delta(\underline{G}(s), \underline{G}^D(s))$. These δ-neighborhoods then sweep out for all s on D broad abstract Nyquist loci about the Γ_i for each $i = 1,\ldots,k$.

Theorem 4.1 is merely one of several theorems along these lines that can be deduced. The particular notion of $G^D(s)$ (the block diagonal portion of G(s)) representing the case of complete decoupling is of course not nescessary for decoupling to be evident. Indeed, if G(s) is block upper triangular then the system is decoupled with respect to the partitioning of F.

Let

$$G^{ri}(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1k}(s) \\ G_{i-1,1}(s) & G_{i-1,2}(s) & \cdots & \\ 0 \quad 0 & \cdots & G_{ii}(s) & 0 \cdots 0 \\ G_{i+1,1}(s) & G_{i+1,2}(s) & \cdots & \\ G_{k1}(s) & \cdots & & G_{kk}(s) \end{pmatrix} \quad (4.9)$$

where the i^{th} block row of G(s) retains its diagonal block -- all others being set to zero. Associated with this define

$$\underline{G}^{ri}(s) = \ker[G^{ri}(s), -I_p]. \quad (4.10)$$

Theorem 4.2: With Γ_G as in theorem 4.1. If

$$\gamma(\underline{G}_i(s), \underline{F}_i) \quad > \quad \delta(\underline{G}(s), \underline{G}^{ri}(s)) \quad (4.11)$$

for each $i = 1,\ldots,k$ and for all s on D then

$$\rho \quad = \quad \sum_{i=1}^{k} \rho_i .$$

Proof: Follows along lines of theorem 4.1.

Clearly this gives distinct fuzzy generalized Nyquist loci for each of the subsystem MIMO feedbacks.

4.2 Computational Considerations for the Weak Coupling Measure

If we assume that frequency response data $G(j\omega)$ is available on a significant portion of the imaginary axis then the stability of a decentralized control system can be determined using either theorem 4.1 or theorem 4.2 if the appropriate weak coupling inequalities (either (4.4) or (4.11)) are satisfied. Indeed, in this case if the individual subsystems

$$H_i(s) = G_{ii}(s) [I_m + F_i G_{ii}(s)]^{-1}$$

are stable for each $i = 1,\ldots,k$ then the overall stability of the decentralized control system follows. Thus computational considerations focus on evaluation of either (4.4) or (4.11). Now computation of the quatities $\delta(.,.)$ and $\gamma(.,.)$ can be done in a numerically stable manner following the procedure suggested in section 3.

For example to compute the quantity $\delta(\underline{G}(s), \underline{G}^D(s))$ for some s on D we perform the following simple procedure:

step 1: Obtain a unitary basis for $\underline{G}(s)$. This is done by obtaining a QR factorization as

$$\begin{pmatrix} I_m \\ G(s) \end{pmatrix} = [Q_G, Z] \begin{pmatrix} R \\ 0 \end{pmatrix}$$

the columns of Q_G are an orthonormal basis for $\underline{G}(s)$.

Remark: The QR factorization outlined here can be performed using a numerically stable algorithm involving the use of Householder reflections to compute the transformations Q_i. This has been implemented efficiently in LINPACK routine CQRDC [14].

step 2: Repeat step 1 to find a basis for $\underline{G}^D(s) = \ker[G^D(s), -I_p]$ as range($Q_G D$).

step 3: Following corollary 3.8 compute the maximum singular value of

$$[I_{p+m} - Q_G Q_G^*] Q_{GD} .$$

Remark: The product of unitary matrices can be obtained in a numerically stable way. Then application of a standard algorithm can provide the required singular value. The routine CSVDC is available in LINPACK for computing these quantities [14].

5.0 Conclusions

In this paper we have provided a basic method for testing the weak coupling hypothesis for large scale systems in terms of a frequency dependent gap-metric. Other researchers have employed weak coupling notions for the stability analysis of decentralized control systems, [1]-[3] but our methods concentrate on providing frequency dependent shaping requirements for linear, time-invariant systems.

The connection between the gap-metric and the minimum frequency dependent singular value of the matrix return difference is not yet clear, however the computational procedures suggested in section 3 and the connection between the gap-metric and the principal angles between subspaces allows more information about robustness to be obtained in this procedure. In particular "sensitive directions" in the space of transfer function matrices with respect to stability can be expressed in terms of the principal vectors for certain principal angles. Computational procedures and applications of these ideas will be discussed in some forth coming work.

Of significance to the current problem is that one can extend the notion of sensitive directions in the analysis of robustness of MIMO feedback to the case of F = block diag$\{F_1,\ldots,F_k\}$; i.e., decentralized control. Depending on the notion of weak coupling employed we can as in theorem 4.2 investigate robustness with respect to variations of the local controllers F_i for $i = 1,\ldots k$. We feel this aspect will be most significant in a computer-aided design environment for control of large scale systems.

Clearly the implementation of these ideas will require intensive computer computations. In this regard the use of unitary bases representing the subspaces is of course significant since the indicated computations can be done without significant loss of precission. Of course the computation indicated in (4.4) or (4.11) for determining weak coupling may be indeed costly by comparison with for instance the BDD methods employed in [5] however we believe the additional information available from principal vector analysis for design. Current research efforts along the lines of providing connections with the gap-metric and other approximate methods for determining weak coupling (such as BDD methods) will be reported elsewhere.

6.0 References

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