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BLOCK DIAGONAL DOMINANCE AND DESIGN OF DECENTRALIZED COMPENSATORS

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Abstract

An extension of Rosenbrock's design method for linear multivariable systems is presented, based on the concept of block diagonal dominance for rational transfer function matrices. The technique allows independent design of compensators for low-interacting subsystems. The flexibility of the method with respect to partitioning and measures of gain leads to improved estimates for overall system stability under decentralized compensator design. Various new directions and extensions suggested by our methods are discussed and examples illustrate the theory.

1. Introduction

The current literature on control of large scale systems is dominated by technqiues for model simplification and order reduction [6]. A natural trend in this area is the decentralized control philosophy. Our approach here is to study the natural partitioning of a system into subsystems with "low" interaction; and to develop sufficient conditions for stability of the closed loop system using decentralized control for linear, time-invariant systems. We describe here the systems by matrix transfer functions.

During the present decade transfer function (or frequency domain) design methods have been developed [1]-[5], that offer to the designer some substantial advantages over state space methods. Primarily, these methods have been developed for centralized compensator design. The method proposed here is frequency domain design method for decentralized compensator design. As such (and primarily due to its intrinsic flexibility) the method leads to a natural, frequency dependent, treatment of interacting subsystems and suggests natural partitions (provided such exist) of large systems.

The general multivariable control system is shown in figure 1 below.

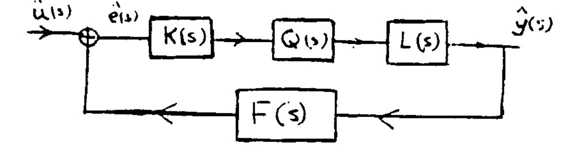

Fig. 1. The general closed-loop system.

Here $Q(s)$ is the plant transfer function matrix and is assumed pxm, $K(s)$ is the input compensator mxn, $L(s)$ the output compensator nxp and $F(s)$ the feedback compensator nxn. Letting,

$$G(s) = L(s)Q(s)K(s) \tag{1.1}$$

for the forward loop transfer function, the closed loop transfer function is

$$\begin{aligned} H(s) &= [I_n + G(s)F(s)]^{-1}Q(s) \\ &= G(s)[I_n + F(s)G(s)]^{-1}. \end{aligned} \tag{1.2}$$

Defining the inverse matrices

$$\hat{G} \triangleq G^{-1}$$
$$\hat{H} \triangleq H^{-1} \tag{1.3}$$

(provided they exist) we have the inverse relationship

$$\hat{H}(s) = F(s) + \hat{G}(s). \tag{1.4}$$

For obvious practical reasons Rosenbrock in [1,2] searched for ways to design feedback compensators of the simple form $F(s) = \mathrm{diag}\{f_i\}$. In [1,2] Rosenbrock and his coworkers developed a practical technique for compensator design called Inverse Nyquist Array method. This well known method can be effectively supported by interactive computer codes and has been rather extensively used to design compensators for industrial processes[2].

Of primary importance in this method is the concept of a diagonal dominant matrix on a contour D of the complex plane. A rational nxn matrix $Z(s)$ is diagonally dominant on the contour D if $z_{ii}(s)$ has no pole on D, $i = 1, 2, \ldots, k$ and for each s on D

$$|z_{ii}(s)| - \sum_{\substack{j=1 \\ j\neq i}}^{k} |z_{ij}(s)| > 1, i=1, 2, \ldots n$$

or

$$|z_{ii}(s)| - \sum_{\substack{j=1 \\ j\neq i}}^{k} |z_{ji}(s)| > 0, i=1, 2, \ldots n \tag{1.5}$$

The technique consists then in designing first the input compensator $K(s)$ ($L(s) = I_n$ usually) so that $\hat{G}(s)$ is diagonally dominant on a relatively large part (with respect to the poles and zeros of G(s) of the imaginarty axis ($0 \le \omega \le \omega_{max}$). This then allows by use of Gerschgorin estimates [2] for eigenvalues, the design of feedback gains f_i for each loop independently, guaranteeing stability of the overall system when all loops are closed. Furthermore, after selecting the gains, using the Ostrowski refinement of Gerschgorin's theorem, "fuzzy" inverse Nyquist plots for each loop (in the sense of an approximation) are derived and thus the stability of the design evaluated. The technique is extremely useful since most of the work can be effectively performed by interactive computer code and graphics.

Our method concerns decentralized compensator design and is reminiscent of Rosenbrock's Inverse Nyquist Array method. By decentralized compensation we understand that the feedback compensator transfer matrix has the partitioned form.

$$F(s) = \mathrm{diag}[F_1(s), F_2(s), \ldots, F_m(s)] \quad (1.6)$$

where $F_i(s)$ is $k_i \times k_i, \sum_{i=1}^{m} k_i = n$. Thus the n vectors $\hat{e}(s), \hat{u}(s), \hat{y}(s)$ are partitioned conformally into m subvectors each. Similarly, the open loop matrix transfer function G(s) is partitioned in m^2, $k_i \times k_j$ blocks $G_{ij}(s)$. As a result the inverse closed loop matrix transfer function is also partitioned

$$\hat{H}_{ij}(s) = \begin{cases} \hat{G}_{ii} + F_i(s), & i=j \\ \hat{G}_{ij}(s), & i \neq j \end{cases} \quad (1.7)$$

We are thus lead to m separate compensator design problems and we want to develop conditions that guarantee desired performance of the overall system. The latter is equivalent to assuring "low" subsystem interaction. There are two main reasons that render decentralized compensation desirable in large scale systems: (a) the performance requirements on each of the m subsystems may be of completely different nature and, therefore, may require different design approaches; (b) the reduction in the dimension of the overall problem leads to significant reduction in the computational complexity of the design procedure.

We utilize here results on block diagonal dominance and the related "block Gerschgorin Theorems" [7]-[8], to guarantee "low" subsystem interaction. The resulting design approach is a three step process. First, a test is performed on the open loop plant transfer function matrix. If the test is satisfied a decoupled design approach is pursued where the "diagonal" subsystems are assumed to be non-interacting. Complete freedom is available here as to design technique for the parallel decoupled subsystems designs. Finally, a test similar to that performed in the first step is performed on a matrix constructed of the plant and feedback matrices in a familiar way. If this test is satisfied then the closed loop system stability performance is satisfactory.

It is worth emphasizing that in this paper we have utilized only the results of Feingold and Varga [7]. The results in [8]-[11] lead to several alternative methods for decentralized compensator design which will be reported elsewhere. Finally, (as we shall see) the flexibility of our method has not been fully utilized here in order to provide optimal estimates of overall system performance. Work is in progress on this and related numerical aspects and will be reported elsewhere.

2. Block Diagonal Dominance and its Consequences

The work of Gerschgorin (which plays a fundamental role in the Inverse Nyquist Array method) on estimates for the eigenvalues of a square complex matrix has been generalized by Feingold and Varga [7] and in a slightly different way by Fiedler and Ptak [8] to partitioned matrices. These results not only provide a variety of alternative estimates but actually can lead to tighter estimates than the traditional Gerschgorin circles [7]. Following [7] let A be an $n \times n$ complex matrix partitioned into m^2 submatrices, so that A_{ij} is $k_i \times k_j$, $1 \le i, j \le m$, $\sum_{i=1}^{m} k_i = n$. We introduce vector norms on the subspaces X_i, $i=1, \ldots, m$ of C^n implied by this decomposition, where (and this is very important) different subspaces are allowed different norms. We denote all vector norms by $|\cdot|$ for simplicity of notation, letting the vector indicate which one of the norms is applied (by the subspace where the vector belongs). We also consider the induced matrix norms

$$\| A_{ij} \| = \sup_{\substack{x \in X_j \\ x \neq 0}} \frac{|A_{ij} x|}{|x|} \quad (2.1)$$

and infimum or reciprocal norms

$$\lfloor A_{ij} \rfloor = \inf_{\substack{x \in X_j \\ x \neq 0}} \frac{|A_{ij} x|}{|x|} \quad (2.2)$$

Clearly if A_{ii} is nonsingular

$$\lfloor A_{ii} \rfloor = \| A_{ii}^{-1} \|^{-1} \quad (2.3)$$

We then have as in [7]

Definition 2.1: The partitioned matrix A is block diagonally dominant with respect to this partition if

(i) the diagonal submatrices, A_{ii}, are nonsingular and (ii) either

$$\| A_{ii}^{-1} \|^{-1} > \sum_{\substack{j=1 \\ j\neq i}}^{m} \| A_{ij} \| \text{ , } i = 1, 2, \ldots m \tag{2.4}$$

$$\left(\text{or } \| A_{ii}^{-1} \|^{-1} > \sum_{\substack{j=1 \\ j\neq i}}^{m} \| A_{ji} \| \text{ , } i = 1, 2, \ldots m.\right)$$

Clearly, when $k_i = 1$, $i = 1, \ldots m$ this definition reduces to the usual dominance condition [1, p. 142]. It is not difficult to show that if the partitioned matrix A is block diagonally dominant it is nonsingular [7, Th. 1], which then leads to the following generalization of Gerschgorin's theorem [7, Th. 2]:

Theorem 2.2: For a matrix A partitioned as above, each eigenvalue λ satisfies

$$\|(A_{ii} - \lambda I_{k_i})\| \leq \sum_{\substack{j=1 \\ i\neq j}}^{m} \| A_{ij} \| \tag{2.5}$$

$$\left(\text{or} \leq \sum_{\substack{j=1 \\ i\neq j}}^{m} \| A_{ji} \|\right)$$

for at least one $1 \leq i \leq m$.

Thus Theorem 2.2 describes inclusion regions for the spectrum of A analogous to the well known Gerschgorin circles. Further details and examples on the regions provided by (2.5) can be found in [7]. It is important to emphasize the flexibility provided by this "Block Gerschgorin Theorem": a) the inclusion regions depend on the vector norms used on the subspaces; b) the inclusion regions depend on the partitioning of the matrix. Both facts suggest considering various alternative combinations in order to provide optimum estimates (i.e., tight regions). This important characteristic of the "Block Gerschgorin Theorem" (the classical Gerschgorin Theorem does not provide such flexibility) has been exploited successfully in [7]-[9].

Let $\mathcal{G}_i$ (or $\mathcal{G}_i'$) be the ith "Gerschgorin set" of complex numbers λ satisfying (2.5).

$$\mathcal{G}_i = \{\lambda ; \| A_{ii} - \lambda I_{ki} \| \leq \sum_{\substack{j=1 \\ j\neq i}}^{m} \| A_{ij} \|$$

or

$$\mathcal{G}_i' = \{\lambda ; \| A_{ii} - \lambda I_{ki} \| \leq \sum_{\substack{j=1 \\ i\neq j}}^{m} \| A_{ji} \|$$

Clearly these sets are closed and bounded and thus compact. Then all eigenvalues of A are in

$$\mathcal{G} = \bigcup_{i=1}^{m} \mathcal{G}_i \quad \left(\text{or } \mathcal{G}' = \bigcup_{i=1}^{m} \mathcal{G}_i'\right)$$

Furthermore, the eigenvalues of A_{ii} are in $\mathcal{G}_i$. A useful result towards characterizing how many eigenvalues of A are included in a subset of G is given by the following result [7].

Theorem 2.3. If the union $\mathcal{H} = \bigcup_{j=1}^{\ell} \mathcal{G}_{p_j}$, $1 \leq p_j \leq m$, of ℓ Gerschgorin sets is disjoint from the remaining $m - \ell$ Gerschgorin sets for a partitioned matrix A, then $\mathcal{H}$ contains precisely $\sum_{j=1}^{\ell} k_{p_j}$ eigenvalues of A.

We proceed now to derive the main result of this paper, on which several design procedures can be based. First we need:

Definition 2.4: Let A(s) be an nxn rational matrix partitioned as above, and D a closed elementary contour in C. Then A(s) is said to be block diagonally dominant on D if: (i) $A_{ii}(s)$ has no pole on D, $i = 1, \ldots m$ and (ii) A(s) is block diagonally dominant for all s on D (Definition 2.1).

We then have the following generalization of Rosenbrock's result [1, Th. 1.9.4].

Theorem 2.4: Let A(s) be an nxn rational matrix partitioned as above, which is block diagonally dominant on a closed elementary contour D in the complex plane. As s traces D once clockwise, let det A(s) map D into the curve Γ_A which encircles the origin N_A times clockwise, det $A_{ii}(s)$ map D into Γ_i which encircles the origin N_i times clockwise $i = 1, 2, \ldots, m$. Then

$$N_A = \sum_{i=1}^{m} N_i \tag{2.6}$$

Proof: The proof generalizes appropriately the proof in [1]. Since by assumption (Definition 2.4 (i)) $A_{ii}(s)$ has no pole on D, it is finite on D, and so is det $A_{ii}(s)$. By block diagonal dominance (Definition 2.1 (i)) det $A_{ii}(s)$ has no zero on D so $\| A_{ii}^{-1}(s) \|^{-1}$ is finite on D. Therefore, from (2.4) $\| A_{ij}(s) \|$ must be finite on D $1 \leq i, j \leq m$, $i \neq j$. So there are no poles of $A_{ij}(s)$ on D, $1 \leq i, j \leq m$. Moreover, by block diagonal dominance A(s) is nonsingular on D so there is no zero of det A(s) on D. Let $A(\alpha, s)$ be the partitioned matrix.

$$A(\alpha, s) = \begin{cases} A_{ii}(\alpha, s) = A_{ii}(s) \\ A_{ij}(\alpha, s) = \alpha A_{ij}(s), \; i \neq j \end{cases} \tag{2.7}$$

where $0 \leq \alpha \leq 1$.
Then every element of $A(\alpha, s)$ is finite on D and therefore, $\det A(\alpha, s)$ is also finite on D. Let

$$\beta(\alpha, s) = \frac{\det A(\alpha, s)}{\prod_{i=1}^{m} \det A_{ii}(s)} \tag{2.8}$$

and note that $\beta(0, s) = 1$. Let $\beta(1, s)$ map D into Γ_β. For each s on D, $\beta(\alpha, s)$ defines a continuous curve joining $\beta(0, s) = 1$ and the point of Γ_β corresponding to s. We will be done if we show that Γ_β does not encircle the origin. Assume the contrary. Then there exists some α, $0 \leq \alpha \leq 1$, such that for some s on D, $\beta(\alpha, s) = 0$. Then from (2.8) $\det A(\alpha, s) = 0$. However, since $A(s)$ is block diagonally dominant on D and $0 \leq \alpha \leq 1$, $A(\alpha, s)$ is also block diagonally dominant on D and therefore, nonsingular, thus contradiction. Then from (2.8) the number of encirclements of the origin by Γ_β is

$$0 = N - \sum_{i=1}^{m} N_i$$

and this concludes the proof.

Clearly, we can test graphically whether or not a given rational matrix is block diagonally dominant on a curve D. From (2-5) it follows that the graphical test consists of plotting for every s on D the Gerschgorin sets $\mathcal{G}_i$, $i = 1, \dots, m$ (or $\mathcal{G}'_i$, $i = 1, \dots m$) and testing if the resulting generalized Gerschgorin bands include the origin. If the origin is not included the given transfer function matrix is block dominant on D. The significant implications of our previous remarks on the flexibility of trying various partitions or various norms become apparent. In particular, we single out the following conclusions: a) tighter estimates on dominance may be obtained; b) decompositions "natural" to the frequency characteristics of a given system may be achieved; c) improved stability estimates may be given even for the standard case [1]. Of course, the computational complexity of the procedure is also influenced by these choices.

Since the computation of the sets $\mathcal{G}_i(s)$ (or $\mathcal{G}'_i(s)$) for each s on D may be a cumbersome computational task, simpler tests for block diagonal dominance can be quite useful. To test for dominance we only need to know for each s the quantities

$$\rho_{A,i}(s) = \min_{\lambda \in \mathcal{G}_i(s)} |\lambda| \quad , \; i = 1, \dots m$$

$$\rho'_{A,i}(s) = \min_{\lambda \in \mathcal{G}'_i(s)} |\lambda| \quad , \; i = 1, \dots, m \tag{2.9}$$

where obviously we use min instead of inf. since $\mathcal{G}_i(s), \mathcal{G}'_i(s)$, $i = 1, \dots, m$ are compact. It is then clear that block diagonal dominance is equivalent to $A_{ii}(s)$, $i = 1, \dots m$, having no poles on D and being nonsingular on D and

$$\rho_{A,D} = \min_{s \in D} \left\{ \max \left[\min_{i \in [1,m]} \rho_{A,i}(s) \, , \, \min_{i \in [1,m]} \rho'_{A,i}(s) \right] \right\} > 0. \tag{2.10}$$

Thus, (2.10) is a proper generalization of Rosenbrock's result [1, p. 143, eg. (5.4)]. It is worth emphasizing here that the generalized Gerschgorin bands resulting from block diagonal dominance considerations are not graphically as useful as the Gerschgorin bands appearing in standard diagonal dominance considerations [1], because they do not convey <u>directly</u> usable information for the choice of compensators. Consequently, the important quantities are $\rho_{A,i}(s)$, $\rho'_{A,i}(s)$, $\rho_{A,D}$ which allow (or not) (2.6) to be valid and furthermore, as we shall see later, provide some guidance for compensator selection.

It is easy to see that for any square matrix B

$$\| B - \lambda I \| + \|B\| \geq |\lambda| \geq \|B\| - \|B - \lambda I\| \tag{2.11}$$

and therefore if we consider the tori in the complex plane

$$\mathcal{T}_i(s) = \left\{ \lambda \in C : \|A_{ii}(s)\| - \sum_{j \neq i} \|A_{ij}(s)\| \leq |\lambda| \leq \|A_{ii}(s)\| + \sum_{j \neq i} \|A_{ij}(s)\| \right.$$

$$\mathcal{T}'_i(s) = \left\{ \lambda \in C ; \|A_{ii}(s)\| - \sum_{j \neq i} \|A_{ji}(s)\| \leq |\lambda| \leq \|A_{ii}(s)\| + \sum_{j \neq i} \|A_{ji}(s)\| \right. \tag{2.12}$$

for each s on D we have the inclusions (from (2.5))

$$\mathcal{G}_i(s) \subseteq \mathcal{T}_i(s), \mathcal{G}'_i(s) \subseteq \mathcal{T}'_i(s), \; i = 1, \dots m. \tag{2.13}$$

Clearly then A(s) is diagonally dominant on D iff

$$d_{A,D} = \min_{s \in D} \left\{ \max \left[\min_{i \in [1,m]} d_{A,i}(s) \, , \, \min_{i \in [1,m]} d'_{A,i}(s) \right] \right\} > 0, \tag{2.14}$$

where

$$d_{A,i}(s) = \|A_{ii}(s)\| - \sum_{\substack{j=1 \\ j \neq m}}^{m} \|A_{ij}(s)\|$$

$$d'_{A,i}(s) = \|A_{ii}(s)\| - \sum_{\substack{j=1 \\ j \neq m}}^{m} \|A_{ji}(s)\| \quad . \tag{2.15}$$

Various considerations indicate that the quantities $d_{A,D}, d_{A,i}(s), d'_{A,i}(s)$ are rather tight estimates of $\rho_{A,D}, \rho_{A,i}(s), \rho'_{A,i}(s)$. In particular if $A_{ii}(s), i = 1, \dots m$, are normal the two sets of parameters measuring dominance coincide since in this case each $\mathcal{G}_i(s)$ (or $\mathcal{G}'_i(s)$) is a union of a finite number of disks.

3. Application to Decentralized Feedback Compensation

In this section we develop our design method based primarily on the result of Theorem 2.4. In discussing stability we follow the conventions of Rosenbrock [1, pp. 1-27]. Thus the zeros (resp. poles) of a matrix transfer function G are the zeros of all numerator (resp. denominator) polynomials in the McMillan form of G. The poles of the system are the zeros of detT(s) in any given polynomial matrix representation of the system with transfer function G(s).

$$
\begin{aligned}
T(s)\hat{x}(s) &= U(s)\hat{u}(s) \\
\hat{y}(s) &= V(s)\hat{x}(s) + W(s)\hat{u}(s) \qquad (3.1) \\
G(s) &= V(s)T^{-1}(s)U(s) + W(s).
\end{aligned}
$$

When $T(s) = (sI_n - A)$ the poles of the system are the eigenvalues of A. The system (3.1) is asymptotically stable if all poles of the system are in the open left half plane (OLH).

From (1.2)

$$
\det\left[I_n + G(s)F(s)\right] = \det\left[I_n + F(s)G(s)\right] = \frac{\det G(s)}{\det H(s)} = \frac{\det \hat{H}(s)}{\det \hat{G}(s)} \quad , \qquad (3.2)
$$

a well known relationship which makes transparent the role played by the determinant of the return difference matrix $I + G(s)F(s)$ in multivariable stability considerations. We further assume that GF is strictly proper. Then [1, p. 135], [12], if G is proper H is also proper. In the sequel we assume that the given factorizations of G, F are

$$
\begin{aligned}
G(s) &= N(s)D^{-1}(s) \\
F(s) &= N_F(s)D_F^{-1}(s) \qquad (3.3)
\end{aligned}
$$

without making any explicit assumptions about coprimeness [12] or least order [1]. Then following [1, p. 141] to test stability of the closed loop system we choose a contour D consisting of the part of imaginary axis $[-iR, iR]$ together with a semicircle of radius R in the right half plane. R is chosen large enough to insure that D includes all zeros of $\det(I_n + GF) \cdot \det D \cdot \det D_F$ in the closed right half plane (CRH), with the familiar left-half plane indentations for imaginary zeros. Suppose the open loop system has p_o poles in CRH (i.e., zeros of $\det D \cdot \det D_F$ there). Let $\det(I+GF)$, $\det \hat{G}$, $\det \hat{H}$ map D into curves Γ_{RD}, $\hat{\Gamma}_G$, $\hat{\Gamma}_H$ which encircle the origin clockwise N_{RD}, $\hat{N}_G$, $\hat{N}_H$ times. Then we have the well known [1, p. 141, Th. 4.1, Coroll]

Theorem 3.1: The closed loop system shown in Fig. 1 and described by (3.2) is asymptotically stable if and only if

(a) $N_{RD} = -p_o$

or equivalently

(b) $\hat{N}_G - \hat{N}_H = p_o$

We can now give a series of results (using Theorems 2.4 and 3.1) which are generalizations of Rosenbrock's results in [1, 3.5, 3.6]. In the sequel the curve D and p_o are as above, the open loop system poles in CRH.

Theorem 3.2: Suppose H and G are both block diagonally dominant on D. Let $\det G_{ii}$, $\det H_{ii}$ map D into $\Gamma_{G,i}$, $\Gamma_{H,i}$ which encircle the origin $N_{G,i}$, $N_{H,i}$ times $i = 1, \ldots, m$ clockwise. Then the closed loop system is asymptotically stable if and only if

$$
\sum_{i=1}^{m} N_{H,i} - \sum_{i=1}^{m} N_{G,i} = -p_o
$$

Proof: Let det G, det H map D into Γ_G, Γ_H which encircle the origin N_G, N_H times clockwise. Then the closed loop system is asymptotically stable iff $-p_o = N_{RD} = N_G - N_H$ from (3.2). The result follows from Theorem 2.4.

Theorem 3.3: Suppose that F(s) represents an asymptotically stable compensator (i.e., $\det D_F$ has no zeros in CRH, and that $(F^{-1} + G)$ is block diagonally dominant on D. Let $\det(F_i^{-1} + G_{ii})$ map D into Γ_i which encircles the origin N_i times clockwise $i = 1, \ldots, m$. Then the closed loop system is asymptotically stable if and only if

$$
\sum_{i=1}^{m} N_i = -p_o
$$

Proof: From (3.2) $\det[I_n + GF] = \det[F^{-1} + G]\det F$. The result follows from Theorems 3.1 and 2.4.

We note that computation of the closed loop transfer function is not needed. To test block dominance in Theorem 3.3, our remarks in section 2 apply. A sufficient condition is given by the following corollary. Work on establishing more efficient methods is in progress.

Corollary 3.4: If

$$
\| F_i(s) \|^{-1} > \| G_{ii}(s) \| + \sum_{\substack{j=1 \\ i \neq j}}^{m} \| G_{ij}(s) \| \quad \Big(\text{or} \sum_{j \neq i} \| G_{ji}(s) \|\Big)
$$

or

$$
\| F_i(s) \|^{-1} < \| G_{ii}(s) \| - \sum_{j \neq i} \| G_{ij}(s) \| \quad \Big(\text{or} \sum_{j \neq i} \| G_{ji}(s) \| \qquad (3.4)
$$

$i = 1, \ldots, m$

for all s on D then $F^{-1}+G$ is block diagonally dominant on D.

Proof: Follows from $\|A+B\| \geq \|A\| - \|B\|$ (or $\geq \|B\| - \|A\|$)

This simple corollary leads to the following useful graphical test. In fig. 2 below we plot the curves

$$\|G_{ii}(s)\| + \sum_{\substack{j=1 \\ j\neq m}}^{m} \|G_{ij}(s)\|$$

$$\|G_{ii}(s)\| - \sum_{\substack{j=1 \\ j\neq m}}^{m} \|G_{ij}(s)\| \text{ , for } s = i\omega,$$

$\omega \in [0, \omega_b]$, where ω_b is chosen according to our knowledge about the system and other practical considerations [1, 6.2].

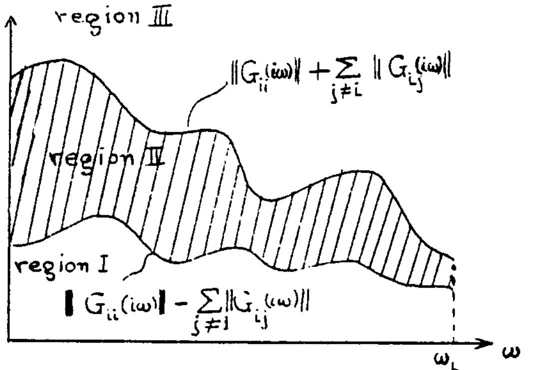

Fig. 2. Graphical interpretation of a test for block dominance of $F^{-1}+G$.

Thenequation (3.4) states that $\|F_i(i\omega)\|^{-1}$ must be in III or $\|F_i(i\omega)\|^{-1}$ in I . It is noted however, that the constraints on F obtained by Corollary 3.4 may be conservative.

For reasons that are well explained in [1] it is convenient to investigate stability of the decentralized controller using the inverse relationships (1.3). We thus have the following results:

Theorem 3.5: Suppose $\hat{G}$. $\hat{H}$ are block diagonally dominant on D. Let det $\hat{G}_{ii}$, det $\hat{H}_{ii}$ map D into $\hat{\Gamma}_{G,i}$, $\hat{\Gamma}_{H,i}$ which encircle the origin $\hat{N}_{G,i}$, $\hat{N}_{H,i}$ times clockwise $i=1,\ldots,m$. Then the closed loop system is asymptotically stable if and only if

$$\sum_{i=1}^{m} \hat{N}_{G,i} - \sum_{i=1}^{m} \hat{N}_{H,i} = p_o$$

Proof: Follows from (3.2), Theorem 3.1 (b) and Theorem 2.4.

Corollary 3.6: If

$$\|F_i(s)\| > \|\hat{G}_{ii}(s)\| + \sum_{\substack{j=1 \\ j\neq i}}^{m} \|\hat{G}_{ij}(s)\| \quad \left(\text{or } \sum_{j\neq i} \|\hat{G}_{ji}(s)\|\right)$$

or

$$\|F_i(s)\| < \|\hat{G}_{ii}(s)\| - \sum_{\substack{j=1 \\ j\neq i}}^{m} \|\hat{G}_{ij}(s)\| \quad \left(\text{or } \sum_{j\neq i} \|\hat{G}_{ji}(s)\|\right)$$

$i=1,\ldots,m,$
for all s on D, then $\hat{H}$ is block diagonally dominant on D.

Proof: From (1.7) and the fact that $\|A+B\| \geq \|A\| - \|B\|$ (or $\geq \|B\| - \|A\|$)

This corollary leads to a graphical test similar to that of corollary 3.4.

Recall that in the standard application of diagonal dominance techniques [1] in compensator design, it is the inverse form that is more useful primarily because an application of Ostrowski's theorem [1, p. 27], provides a reduction on the size of the Gerschgorin bands and thus improved estimates on gain and phase margin of the compensated system [1, 3.6]. We now give similar results in our framework. The first two theorems below are a generalization of Ostrowski's results, to partitioned matrices. First notice that the block diagonal dominance condition (2.4) can be written also as

$$\theta_i \|A_{ii}\| = \sum_{\substack{j=1 \\ j\neq i}}^{m} \|A_{ij}\|, \; i=1,\ldots,m$$
$$\text{(or } \theta_i' \|A_{ii}\| = \sum_{\substack{j=1 \\ j\neq i}}^{m} \|A_{ji}\|, \; i=1,\ldots,m \qquad (3.5)$$

for some $0 \leq \theta_i < 1$ (or $0 \leq \theta_i' < 1$).

Theorem 3.7: If the partitioned matrix A satisfies (3.5), then A has an inverse $\hat{A} = A^{-1}$ which satisfies

$$\|\hat{A}_{ji}\| \leq \theta_j \|\hat{A}_{ii}\|$$
$$\left(\text{or } \|\hat{A}_{ij}\| \leq \theta_j' \|\hat{A}_{ii}\|\right) \qquad (3.6)$$

for $i=1,\ldots,m$, $j=1,2,\ldots,i-1,i+1,\ldots,m$.

Proof: Since A is dominant it is nonsingular. So $\hat{A}$ exists and

$$\sum_{k=1}^{m} A_{jk}\hat{A}_{ki} = 0 \quad , \; j \neq i$$

or

$$\hat{A}_{ji} + A_{jj}^{-1} \sum_{k\neq j}^{m} A_{jk}\hat{A}_{ki} = 0.$$

Now taking norms we have

$$\|\hat{A}_{ji}\| \leq \max_{k\neq j} \|\hat{A}_{ki}\| \sum_{k\neq j} \|A_{jj}^{-1}\| \|A_{jk}\| = \theta_j \max_{k\neq j} \|\hat{A}_{ki}\| \qquad (3.7)$$

due to (3.5). Since (3.7) holds for $j = 1, 2, \ldots, i-1, i+1, \ldots, m$ and $\theta_j < 1$, $\max_{k\neq j}\|\hat{A}_{ki}\| = \|\hat{A}_{ii}\|$ and the result follows.

Theorem 3.8: Let the partitioned matrix A satisfy (3.5). Define

$$\Phi_i = \max_{k\neq i} \theta_k \quad (\text{or } \Phi_i' = \max_{k\neq i} \theta_k')$$

Then $\|\hat{A}_{ii}^{-1} - A_{ii}\| < \theta_i \Phi_i \|A_{ii}\|$ (< or $\theta_i' \Phi_i' \|A_{ii}\|$) (3.8)

for each $i = 1, \ldots, m$.

Proof: Again A is nonsingular; so

$$\sum_{k=1}^{m} A_{ik}\hat{A}_{ki} = I, \quad i = 1, \ldots, m.$$

or

$$(\hat{A}_{ii}^{-1} - A_{ii})\hat{A}_{ii} = \sum_{k\neq i}^{m} A_{ik}\hat{A}_{ki} .$$

Taking norms we have

$$\|(\hat{A}_{ii}^{-1} - A_{ii})\hat{A}_{ii}\| \leq \sum_{k\neq i}^{m} \|A_{ik}\| \|\hat{A}_{ki}\| . \qquad (3.9)$$

However, it is easily seen that $\|AB\| \geq \|B\| \, \|A\|$ and therefore (3.9) implies

$$\|\hat{A}_{ii}^{-1} - A_{ii}\| \leq \sum_{k\neq i}^{m} \|A_{ik}\| \|\hat{A}_{ki}\| / \|\hat{A}_{ii}\|$$

$$\leq \sum_{\substack{k\neq i\\k=1}}^{m} \|A_{ik}\| (\max_{k\neq i} \theta_k) = \theta_i \Phi_i \|A_{ii}\|$$

by (3.5) and the definition of Φ_i.

We now apply theorem 3.8 to the closed loop transfer function matrix H of the system with the decentralized compensator $F(s) = \text{diag}\{F_i(s)\}$, as above. We work with inverse relations and assume that $\hat{H} = F + \hat{G}$ is block diagonally dominant on a curve D. Then the quantities θ_i, θ_i', Φ_i, Φ_i' become functions of s. We can now give the very useful.

Theorem 3.9: Let $\hat{H}(s) = F(s) + G(s)$ be block diagonally dominant on D. Then for each s on D we have

$$\|H_{ii}^{-1}(s) - (F_i(s) + \hat{G}_{ii}(s))\| < \hat{\theta}_i(s)\hat{\Phi}_i(s) \|\hat{G}_{ii}(s)\| < \hat{\theta}_i(s) \|\hat{G}_{ii}(s)\| \qquad (3.10)$$

$$\left(\text{or} < \hat{\theta}_i'(s)\, \Phi_i'(s) \|\hat{G}_{ij}(s)\| < \hat{\theta}_i'(s) \|\hat{G}_{ii}(s)\| \right)$$

for each $i = 1, \ldots, m$. Here $\hat{\theta}_i(s)$, $\hat{\theta}_i'(s)$, $\hat{\Phi}_i(s)$, $\hat{\Phi}_i'(s)$, are computed from (3.5), (3.8) with $A = \hat{H}$.

Proof: Apply theorem 3.8 to $A = \hat{H}$.

We also have the immediate

Corollary 3.10: With the notation of Theorem 3.9

$$\|H_{ii}^{-1}(s) - F_i(s)\| < \|\hat{G}_{ii}(s)\| + \hat{\theta}_i(s)\hat{\Phi}_i(s) \|\hat{G}_{ii}(s)\|$$

or

$$\|H_{ii}^{-1}(s) - F_i(s)\| > \|\hat{G}_{ii}(s)\| - \hat{\theta}_i(s)\Phi_i(s) \|\hat{G}_{ii}(s)\| \qquad (3.11)$$

for each $i = 1, \ldots, m$.

Now $\hat{G}_{ii}$ is the inverse of the open loop transfer function of the ith subsystem. Suppose that the ith feedback compensator is removed, i.e., consider the compensated system with compensator $F(s) = \text{diag}\{F_1(s), \ldots, F_{i-1}(s), 0, F_{i+1}(s), \ldots, F_m(s)\}$. Let us call the transfer function between block input i and block output i, under these circumstances, $H_i(s)$. Now clearly

$$\hat{H}_i(s) + F_i(s) = H_{ii}^{-1}(s) \qquad (3.12)$$

and, therefore, theorem 3.9 and corollary 3.10 provide via generalized Ostrowski bands estimates on the deviation of the ith subsystem inverse transfer function from $\hat{G}_{ii}$, due to the feedback compensators imposed on the other subsystems. That is if we wish to design a compensator for the ith subsystem while the other compensators are fixed we must design it for $H_i(s)$ being the open loop transfer function for that subsystem.

It is this type of considerations that render our results most significant for decentralized compensator design. Space limitation does not allow further discussion on this. Further results on these ideas will be reported elsewhere.

4. An Example

Given the unstable plant

$$G(s) = \left[\begin{array}{cc|cc} \frac{18}{s-6} & \frac{-4.5}{s+3} & 0 & \frac{5}{s+2} \\ \frac{7}{s+4} & \frac{17.5}{s-5} & \frac{5}{s+2} & 0 \\ \hline 0 & \frac{5}{s+2} & \frac{18}{s-6} & \frac{-4.5}{s+3} \\ \frac{5}{s+2} & 0 & \frac{7}{s+4} & \frac{17.5}{s-5} \end{array}\right] \qquad (4.1)$$

it is desired to stabilize the plant with a si le feedback compensator. The standard Gerschgorin bands are shown in Fig. 3 for rows 1 and 2 of G(s). G is not diagonally dominant. The design procedure of Rosenbrock would require at this point the design of series compensators L, K so that LGK is dominant. This step is ad hoc at best. Here we have block diagonal dominance, however. Indeed, in

Fig. 4 we plot the dominance ratio

$$\theta_1(s) = \theta_2(s) = \frac{\| G_{12}(s) \|}{\| G_{11}(s) \|}$$

where we use ℓ_2 norms, and $s = i\omega, \omega \epsilon [0, \omega_b]$ $\omega_b = 25$ rad/s. The margin of dominance (recall corollary 3.4 and Fig. 2) is

$$\zeta = \inf_{s \epsilon D} (\| G_{11}(s) \| - \| G_{12}(s) \|) = 0.372 \ .$$

In Fig. 5 we plot the Gerschgorin bands for $G_{11}(s)$ and we see it is diagonally dominant. A constant diagonal compensator for G_{11} and hence for G_{22} can be designed, where $F_1 = F_2 = \text{diag}\{f_1, f_2\}$ with $f_1^{-1} \leq 1.6, f_2^{-1} \leq 1.8$. But in order to guarantee block diagonal dominance of $F^{-1} + G$ we ask $\| F_1 \|^{-1} = \| F_2 \|^{-1} \leq \zeta$ (corollary 3.4). This results to $\min(f_1, f_2) \geq 2.688$. Thus the choice of $F = \text{diag}\{3.0, 3.0, 3.0, 3.0\}$ is guaranteed to stabilize the plant (4.1) (theorem 3.3).

Acknowledgement

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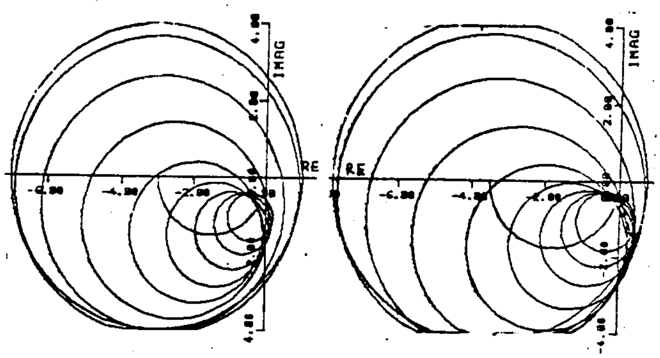

Fig. 3

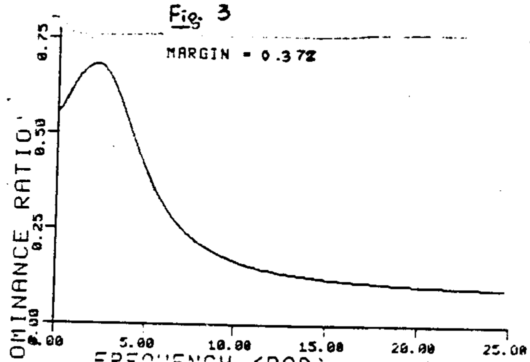

Fig 4

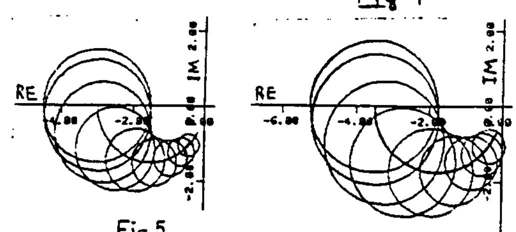

Fig 5