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CONDITIONAL EXPECTATIONS AND FOCK SPACE REPRESENTATIONS IN QUANTUM FILTERING

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ABSTRACT

The role of conditional expectations on von Neumann algebras in the theory of quantum filtering is investigated. The relations characterizing the solution of various quantum filtering problems are interpreted via the notion of conditional expectation. The equations that describe optimal linear quantum filters are represented in the Fock space of analytic functions, and the usefulness of this representation is discussed. Finally some preliminary results on the application of these methods in nonlinear and continuous time filtering are described.

SUMMARY

Quantum detection and estimation problems have been studied heavily during the last few years [1-4], [8-9]. Recently problems of filtering a random signal sequence utilizing quantum mechanical measurements have been investigated [5, 6, 13, 14, 15, 16, 17]. The discrete time filtering problem in general is to estimate x_k, a member of a discrete time signal sequence $\{x_0, x_1, \ldots, x_k, \ldots\}$ (which may be scalar or vector) by optimally (i.e., so that to minimize mean square error) selecting the measurement at time k and the processing (linear or nonlinear) of present and past measurement outcomes. The optical communication setting for this problem is well known [3, 9, 14, 16]. In our previous work [16] and [14, 17] the linear problem for scalar and vector processes has been solved, and various relations characterizing the optimal filter have been established. In this paper some preliminary results in a different direction will be described.

Our approach in [14, 16, 17] was to cast the filtering problem as an abstract optimization problem in a space of operators or positive operator valued measures. Here we discuss the relation of our previous results to conditional expectations on von Neumann algebras. Let H be a Hilbert space and B(H) the algebra of all bounded operators on H. Let A be a subset of B(H) and let A' denote its commutant (i.e., the collection of all bounded operators that commute with operators in A). Then A is a <u>von Neumann algebra</u> if $(A')' = A$. With every subset A of B(H) we can associate the von Neumann algebra generated by it. Let now B be a von Neumann subalgebra of the algebra of bounded operators B(H) of the Hilbert space H and ρ a state on B(H) [2, p. 339] characterized by the density operator $\underset{\sim}{\rho}$. Then [18] [19] the linear mapping ϵ from B(H) onto B is called the

conditional expectation onto B with respect to ρ if

(1) ϵ is an idempotent ($\epsilon^2 = \epsilon$) of norm one

(2) for any uniformly bounded sequence $\{A_n\} \subseteq B(H)$ weakly converging to zero, $\epsilon(A_n)$ weakly converges to zero

(3) $\mathrm{Tr}[\underset{\sim}{\rho}\, \epsilon(A)] = \mathrm{Tr}[\underset{\sim}{\rho} A]$, $A \in B(H)$.

When $\Sigma = \{\rho\}$ is a family of states, we say that a von Neumann subalgebra B is sufficient for Σ if there exists a conditional expectation onto B w.r.t. any $\rho \in \Sigma$ which does not depend on ρ. In the filtering problem we have a family of states $\rho(x_k)$ depending in a measurable way on the signal sequence x_k. The density operators $\underset{\sim}{\rho}(x_k)$ are, as is well known, selfadjoint, nonnegative, with trace one operators on H [7]. The linear filtering problem is to choose measurement at time k with outcome v_k and processing coefficient matrices $C_i(k)$, $i = 0, \dots k$ so that to mimimize

$$\mathrm{MSE} = E\left\{\|x_k - \hat{x}_k\|^2_{R^n}\right\} = E\left\{\|x_k - \sum_{i=0}^{k} C_i(k)\, v_i\|^2_{R^n}\right\} \tag{1}$$

For the scalar signal case the $C_i(k)$ become just scalar coefficients and the measurements are represented by selfadjoint operators V_k on H_k. For the vector case the measurements are represented by positive operator valued measures (p.o.m.) $\underset{\sim}{M}_k$ on H_k [2, 14, 17]. The nonlinear filtering problem is to choose measurement at time k with outcome v_k and a measurable function $f_k(v_0, \dots v_k)$ to minimize

$$\mathrm{MSE} = E\left\{\|x_k - \hat{x}_k\|^2_{R^n}\right\} = E\left\{\|x_k - f_k(v_0, \dots v_k)\|^2_{R^n}\right\} \tag{2}$$

The linear filtering problem has been studied in detail in [14, 16, 17]. Some preliminary results for the nonlinear filtering problem will be presented. One can rewrite (1) [14, 16, 17] as

$$\mathrm{MSE} = \langle \underset{\sim}{\mathfrak{J}}(\cdot\,,\, \underset{\sim}{C}(k)),\, \underset{\sim}{M}_k \rangle_{R^n} \tag{3}$$

where $\underset{\sim}{\mathfrak{J}}(u, \underset{\sim}{C}(k)$ is a nonnegative operator valued function with values trace-class selfadjoint operators, and the symbol in (3) refers to the trace integral [2] of $\mathfrak{J}$ with respect to the p.o.m. $\underset{\sim}{M}_k$ over R^n. Similarly (2) can be rewritten as

$$\mathrm{MSE} = \langle \underset{\sim}{\mathfrak{J}}(\cdot\,,\, f_k),\, \underset{\sim}{M}_k \rangle_{R^n} \tag{4}$$

with a similar interpretation. We will then analyze the relations of our filtering results to conditional expectations on the von Neumann algebra generated by the increments of $\underset{\sim}{\mathfrak{J}}$.

The Fock (or phase) space representation (or P-representation) of quantum fields [20, 21] is a very useful (actually the only one) computational tool. We recast the equations determining the optimal linear filter in Fock space and discuss their importance. Some preliminary results for nonlinear and continuous time filtering, in Fock space representations will be presented also.

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