ENCE 353 Solutions to Midterm 2

Question 1: 15 points

Moment-Area Method. Figure 1 is a front elevation view of a cantilevered beam carrying a single point load P. EI is constant along the beam structure A-B-C-D.

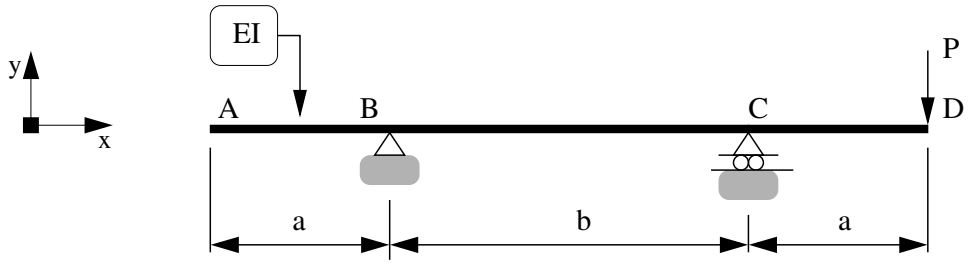


Figure 1. Cantilevered beam carrying a single applied load P.

Part [1a] (2 pts) Draw and label the M/EI diagram.

Sol'n: ...

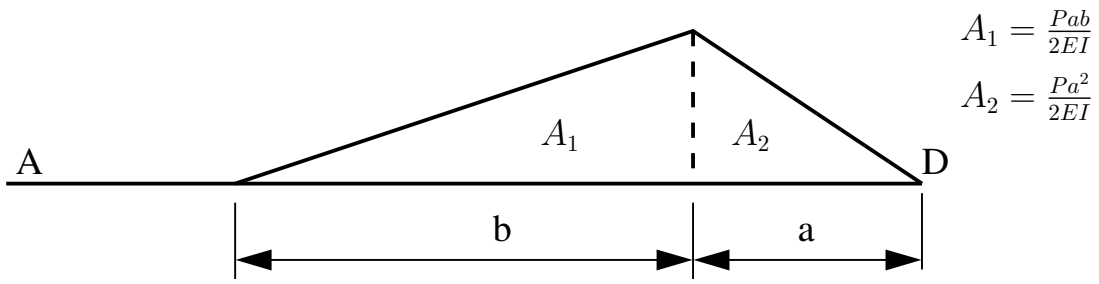


Figure 2. Curvature diagram.

Part [1b] (2 pts) Draw and label the **moment area analysis diagram** (i.e., with rotations, tangents, displacements, etc) for this problem.

Sol'n: See Figure 3.

Part [1c] (3 pts) Use the method of **moment area** to show that the anticlockwise rotation of the beam at B is:

$$\theta_B = \frac{Pab}{6EI} \tag{1}$$

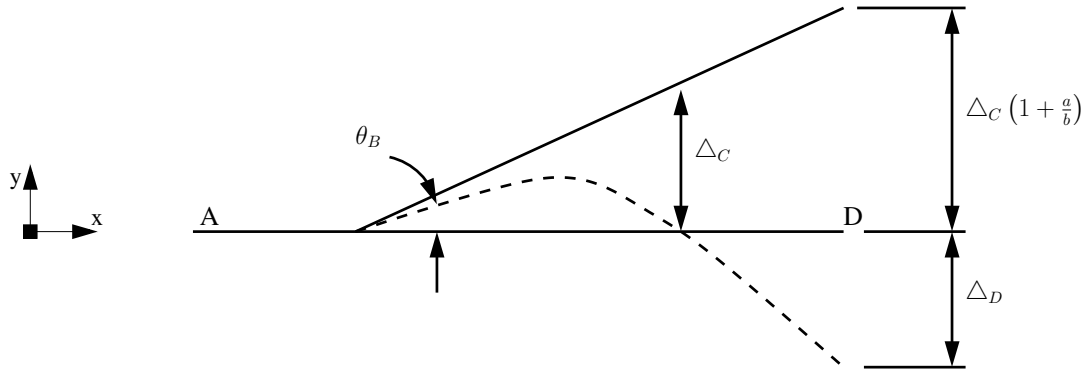


Figure 3. Moment area analysis diagram.

and the clockwise rotation of the beam at C is:

$$\theta_C = \frac{Pab}{3EI}. \tag{2}$$

Sol'n: Apply moment area:

$$\triangle_C = \frac{b}{3}A_1 = \frac{Pab^2}{6EI}, \quad \longrightarrow \quad \theta_B = \frac{\triangle_c}{b} = \frac{Pab}{6EI}. \tag{3}$$

From the first theorem of moment area:

$$\theta_B \text{ (counterclockwise)} + \theta_C \text{ (clockwise)} = A_1 = \frac{Pab}{2EI}. \tag{4}$$

Hence,

$$\theta_C = \frac{Pab}{2EI} - \frac{Pab}{6EI} = \frac{Pab}{3EI}. \tag{5}$$

Part [1d] (3 pts) Use the method of **moment area** to show that the vertical deflection of the beam at points A and D (measured downwards) is:

$$y_A = \frac{Pa^2 b}{6EI} \tag{6}$$

and

$$y_D = \frac{Pa^2}{3EI}(a+b). \tag{7}$$

Sol'n: The downwards deflection at A is given by the rotation at B times the length of the left-most segment, i.e.,

$$\triangle_A = \theta_B a = \frac{Pa^2 b}{6EI}. \tag{8}$$

Next, we can use moment area to find the deflection at D:

$$\triangle_C = A_1 \frac{b}{3}. \tag{9}$$

and

$$\triangle_C(1+\frac{a}{b}) + \triangle_D = A_2(\frac{2}{3}a) + A_1(a+\frac{b}{3}). \tag{10}$$

Combining equations 9 and 10:

$$\triangle_D = \frac{Pa^2}{3EI}(a+b). \tag{11}$$

Part [1e] (5 pts) Using parts [1a] - [1d] as a starting point, show – do not simply assume – that the maximum upwards deflection of the beam occurs at a distance $b/\sqrt{3}$ from B, and that its value is:

$$y_{\text{maximum upward}} = \frac{Pab^2}{9\sqrt{3}EI}. \tag{12}$$

Sol'n: The curvature for section A-B-C of the beam is as follows:

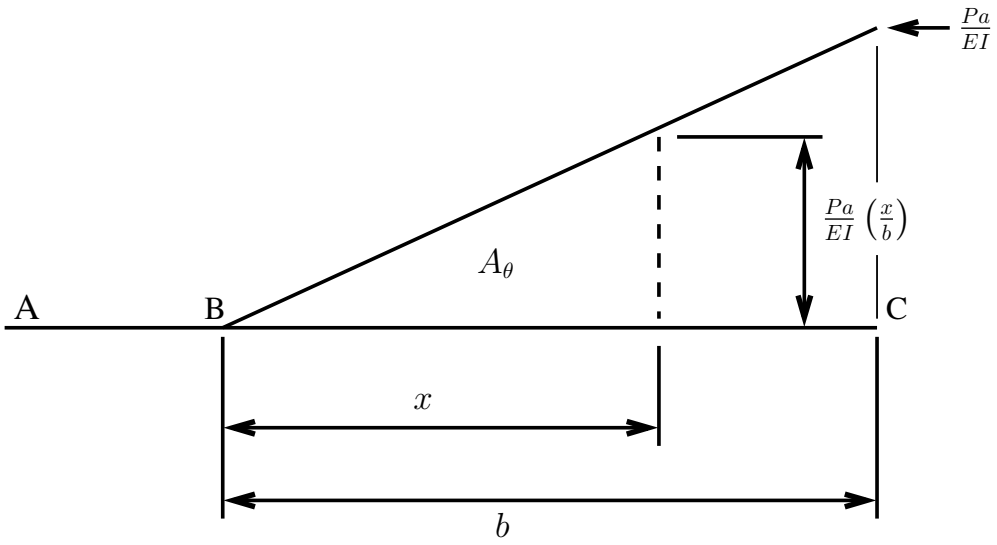


Figure 4. Curvature diagram for A-B-C.

From the geometry:

$$A_\theta = \frac{1}{2}\frac{Pa}{EI}\left(\frac{x}{b}\right)x = \frac{Pax^2}{2EIb}. \tag{13}$$

At the maximum upwards deflection the beam slope = 0. From the first theorem of moment area:

$$\frac{1}{2}\frac{Pax^2}{EIb} = \theta_B = \frac{Pab}{6EI} \quad \longrightarrow x = \frac{b}{\sqrt{3}}. \tag{14}$$

The maximum upwards deflection is given by:

$$y_{\text{maximum upward}} = \theta_B x - A_\theta\left(\frac{x}{3}\right) = \frac{Pab^2}{9\sqrt{3}EI}. \tag{15}$$

Question 2: 10 points

Derive Elastic Curve for Beam Deflection. Figure 5 is a front elevation view of a cantilevered beam carrying a single point load P. EI is constant along the beam structure A-B-C-D.

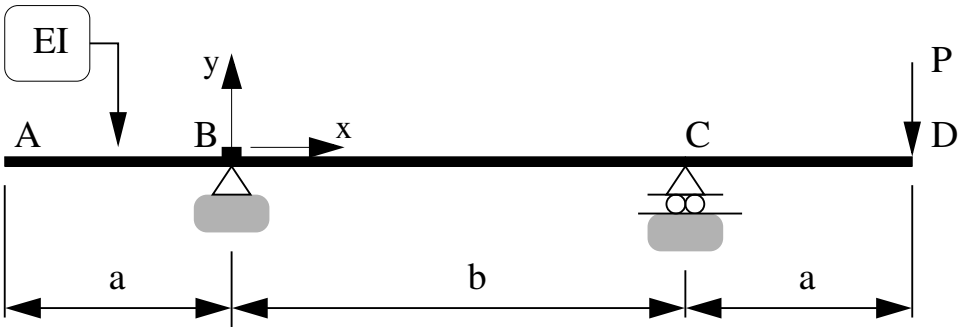


Figure 5. Cantilevered beam carrying a single applied load P.

Part [2a] (5 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{-M(x)}{EI}\right], \tag{16}$$

(notice the minus sign on M(x)) and appropriate boundary conditions, show that that vertical beam deflection along the beam segment B-C is:

$$y(x) = \frac{Pa}{6EIb}\left(b^2 - x^2\right)x. \tag{17}$$

Sol'n: From part [1a],

$$M(x) = -P\left(\frac{a}{b}\right)x. \tag{18}$$

Plugging equation 18 into 16 and integrating twice gives:

$$EIy(x) = -\frac{P}{6}\left(\frac{a}{b}\right)x^3 + Ax + B. \tag{19}$$

Boundary conditions:

$$y(0) = 0, \longrightarrow B = 0. \tag{20}$$

$$y(b) = 0, \longrightarrow A = \frac{Pab}{6}. \tag{21}$$

Combining equations 19 – 21 and rearranging terms:

$$y(x) = \frac{Pa}{6EIb}\left(b^2 - x^2\right) x. \tag{22}$$

Part [2b] (5 pts) Hence (i.e., using your results from part [2a] as a starting point), show that the maximum upwards deflection of the beam occurs at a distance $b/\sqrt{3}$ from B, and that its value is:

$$y_{\text{maximum upward}} = \frac{Pab^2}{9\sqrt{3}EI}. \tag{23}$$

Sol'n: Maximum upwards deflection occurs at:

$$\frac{dy}{dx} = 0 \ \longrightarrow \frac{Pa}{6b}\left(b^2 - 3x^2\right) = 0 \ \longrightarrow x = \frac{b}{\sqrt{3}}. \tag{24}$$

Plugging 24 into equation 22 gives:

$$y(\frac{b}{\sqrt{3}}) = \frac{Pa}{6EIb}\left(b^2 - \frac{b^2}{3}\right)\frac{b}{\sqrt{3}} = \frac{Pab^2}{9\sqrt{3}EI}. \tag{25}$$

Question 3: 15 points

Compute Cable Profile and Tension in a Suspension Bridge. Figure 6 is a front elevation view of a cable span in a suspension bridge.

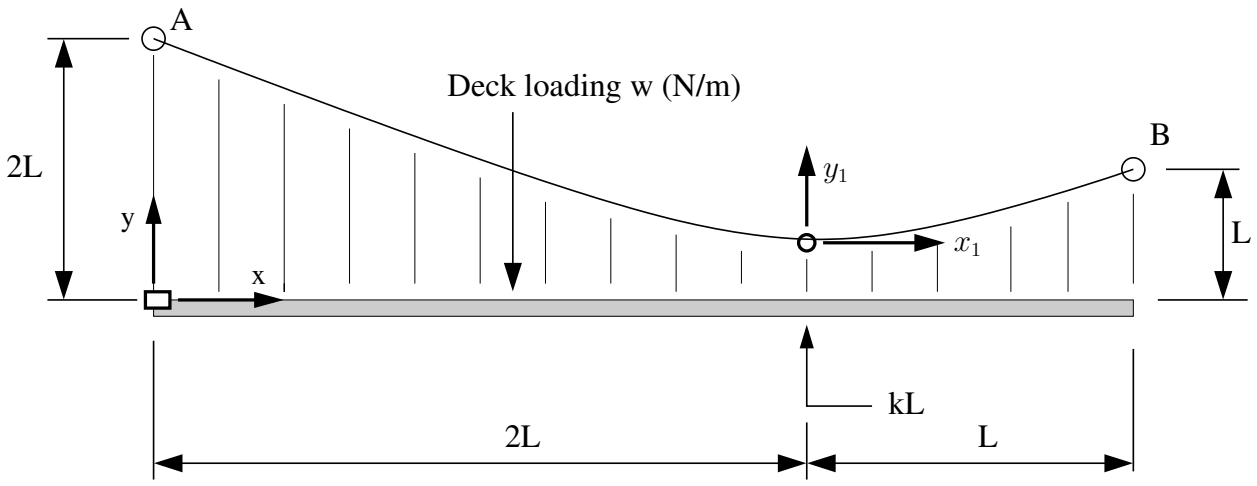


Figure 6. Elevation view of a single cable span in a suspension bridge (not to scale).

The span has length 3L and carries a uniform load w (N/m). The lowest point of the cable profile is kL above the bridge deck.

Part [3a] (4 pts) If the lowest point in the cable profile is at distance 2L from the left-hand side, show that:

$$k = \left[\frac{2}{3}\right]. \tag{26}$$

Sol'n: We can simplify the mathematics by working from a coordinate system (x_1, y_1) as shown in Figure 6. The cable profile is given by:

$$\frac{d^2 y_1}{dx_1^2} = \frac{w}{H} \quad \longrightarrow \quad y_1(x_1) = a x_1^2 \quad \text{where} \quad a = \frac{w}{2H}. \tag{27}$$

From geometry (right-hand side):

$$L(1-k) = AL^2 \tag{28}$$

From geometry (left-hand side):

$$L(2-k) = A(2L)^2 \tag{29}$$

Combining equations 28 and 29 gives:

$$4 - 4k = 2 - k \quad \longrightarrow \quad k = \frac{2}{3}. \tag{30}$$

Part [3b] (4 pts) Hence, show that the horizontal component of cable force is:

$$H = \left[\frac{3}{2}\right] wL. \tag{31}$$

Sol'n: At point B, x_1 = L and y_1 = (1-k)L.

$$(1-k)L = \frac{w}{2H}L^2 \quad \longrightarrow \quad H = \frac{wL}{2(1-k)} = \frac{3}{2}wL. \tag{32}$$

Part [3c] (4 pts) Compute the horizontal and vertical components of cable force at supports A and B.

Sol'n: Recall:

$$y_1(x_1) = \frac{wx_1^2}{2H} \quad \longrightarrow \quad \frac{dy_1}{dx_1} = \frac{wx_1}{H} = \frac{v(x_1)}{H}. \tag{33}$$

The vertical component of cable force is $v(x_1) = wx_1$. At the supports $V_A = 2wL$ and $V_B = wL$.

Finally, we note $V_A + V_B = 3wL$ (i.e., the system is in equilibrium).

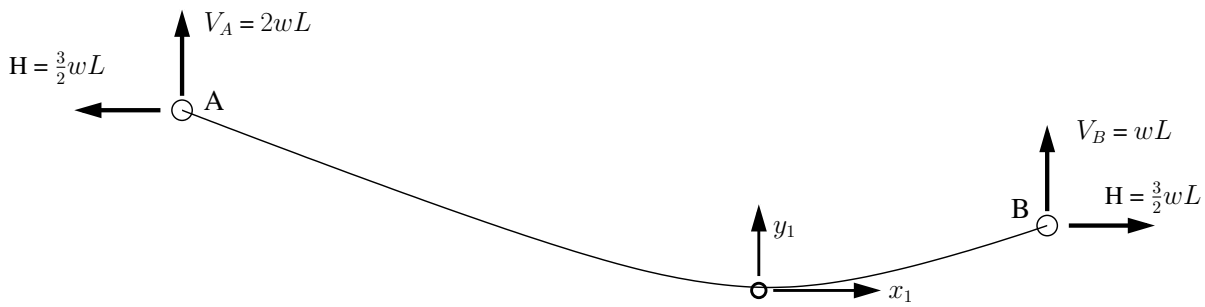


Figure 7. Horizontal and vertical support reactions at A and B.

Part [3d] (3 pts) Draw and label a sketch showing how the tension force in the cable varies along its length. Clearly indicate where the maximum and minimum values of cable tension will occur.

Sol'n:

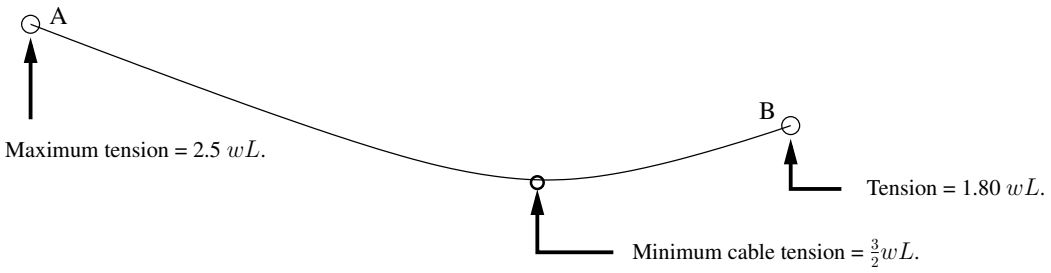


Figure 8. Plot of tension force in the cable.