Problem 1: The cantilever beam structure shown in Figure 1 carries a uniform load w (N/m) along its entire length.

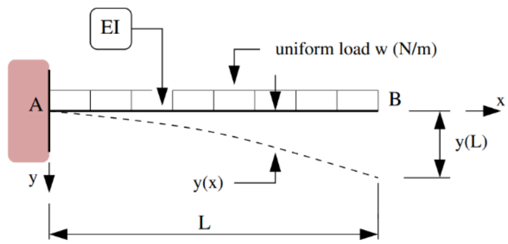


Figure 1: Cantilever beam carrying a uniform load.

The beam is fully fixed at point A and the flexural stiffness EI is constant along the beam. The coordinate system is positioned at point A.

[1a] Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{M(x)}{EI}\right], \tag{1}$$

and appropriate boundary conditions, show that:

$$y(x) = \left(\frac{w}{24EI}\right)\left(6L^2x^2 - 4Lx^3 + x^4\right). \tag{2}$$

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \xrightarrow{\text{Finding } M(x)} M(x) = w(L-x)\frac{L-x}{2} = \frac{w}{2}(L-x)^2$$

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} = \frac{w(L-x)^2}{2EI} = \frac{w}{2EI}(L^2 - 2Lx + x^2) \xrightarrow{\text{Integration}} \frac{dy}{dx} = \frac{w}{2EI}\left(L^2x - Lx^2 + \frac{x^3}{3} + A\right)$$

$$\xrightarrow{\text{Integration}} y(x) = \frac{w}{2EI}\left(\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12} + Ax + B\right) \xrightarrow{\text{Boundary Conditions}} \begin{cases} y(0) = 0 \rightarrow B = 0 \\ \left.\frac{dy}{dx}\right|_{x=0} = 0 \rightarrow A = 0 \end{cases}$$

$$y(x) = \frac{w}{24EI}(6L^2x^2 - 4Lx^3 + x^4)$$

[1b] Using the results of question [1a] as a starting point, compute the support reactions at A and B for the propped cantilever shown in Figure 2.

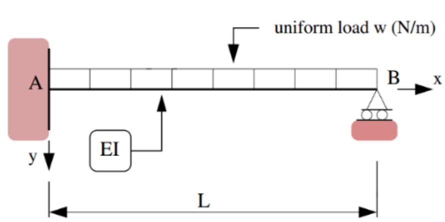
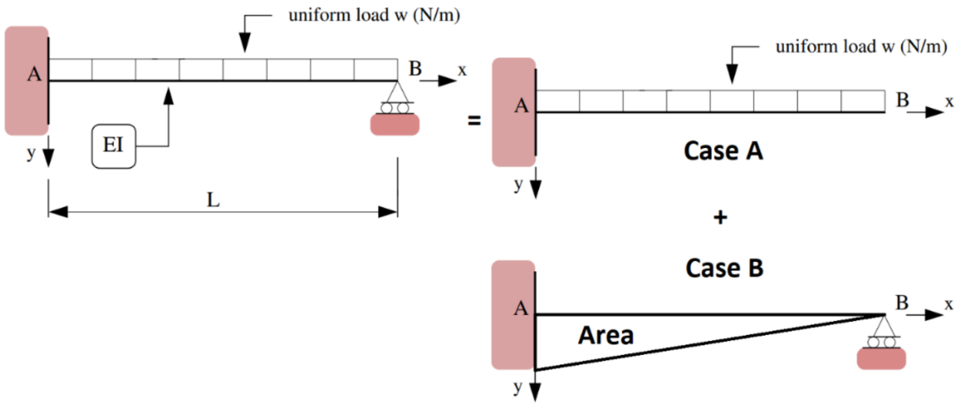


Figure 2: Propped cantilever beam carrying a uniform load.

Using Superposition:



Case A:

$$y(x) = \frac{w}{24EI}(6L^2x^2 - 4Lx^3 + x^4) \rightarrow \Delta_B = y(L) = \frac{w}{24EI}(6L^4 - 4L^4 + L^4) = \frac{w}{24EI}(3L^4)$$
$$= \frac{wL^4}{8EI}\ (downward)$$

Case B: According to the method of moment area:

$$\Delta_B = Area * \bar{x}_B = \left(\frac{1}{2} * L * \frac{V_B L}{EI}\right) * \left(\frac{2}{3}L\right) = \frac{V_B L^3}{3EI}\ (upward)$$

$$\Delta_{B_{total}} = 0 \rightarrow \frac{wL^4}{8EI} = \frac{V_B L^3}{3EI} \rightarrow V_B = \frac{3}{8}wL\ (upward)$$

$$\sum F_y = 0 \rightarrow V_A = \frac{5}{8}wL\ (upward)$$

$$\sum M_A = 0 \rightarrow M_A = \frac{wL^2}{8}\ (CCW)$$

Problem 2: Consider the cantilever shown in Figure 3.

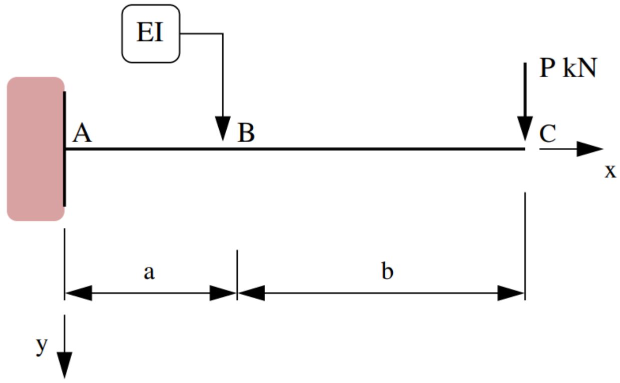
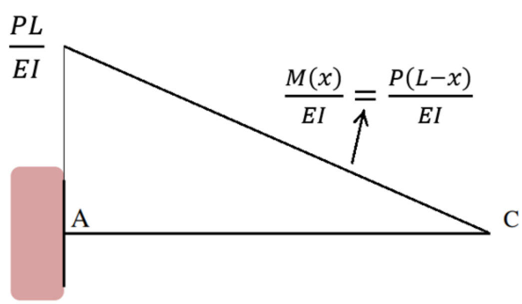


Figure 3: Front elevation view of a cantilever.

The cantilever has constant section properties, EI, along its entire length (a+b). A vertical load P (kN) is applied at point C.

[2a] Use the method of moment area to show that the vertical deflection of the cantilever at point C is:

$$y_C = \frac{P(a+b)^3}{3EI}. \tag{3}$$



$L = a + b\,;\; M(x) = P(L-x) \rightarrow \frac{M(x)}{EI} = \frac{P(L-x)}{EI}$

$\Delta_C = y(L) = Area * \bar{x}_c = \left(\frac{1}{2} * L * \frac{PL}{EI}\right) * \left(\frac{2}{3}L\right) = \frac{PL^3}{3EI} = \frac{P(a+b)^3}{3EI}$

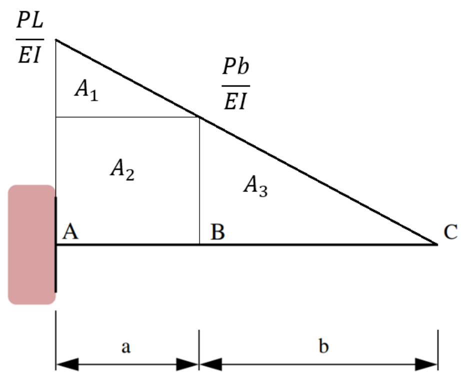
[2b] Use the method of moment area to show that the vertical deflection of the cantilever at point B is:

$$y_B = \frac{Pa^2}{6EI}[3b + 2a]. \tag{4}$$

$\begin{cases} A_1 = \frac{1}{2} * a * \left(\frac{Pa}{EI}\right) = \frac{Pa^2}{2EI} \\ A_2 = a * \left(\frac{Pb}{EI}\right) = \frac{Pab}{EI} \end{cases}$

$\begin{cases} \bar{x}_{1B} = \frac{2}{3}a \\ \bar{x}_{2B} = \frac{a}{2} \end{cases}$

$$\Delta_B = A_1\bar{x}_{1B} + A_2\bar{x}_{2B} = \frac{Pa^2}{2EI} * \frac{2}{3}a + \frac{Pab}{EI} * \frac{a}{2} = \frac{Pa^3}{3EI} + \frac{Pa^2b}{2EI} = \frac{Pa^2}{6EI}(2a + 3b)$$



Now suppose that a roller support is inserted below point B as follows:

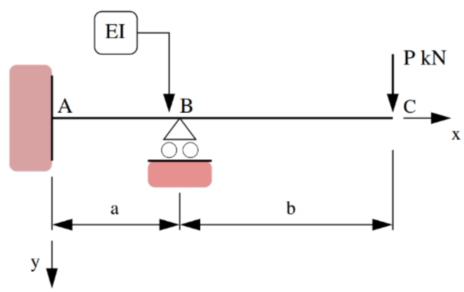
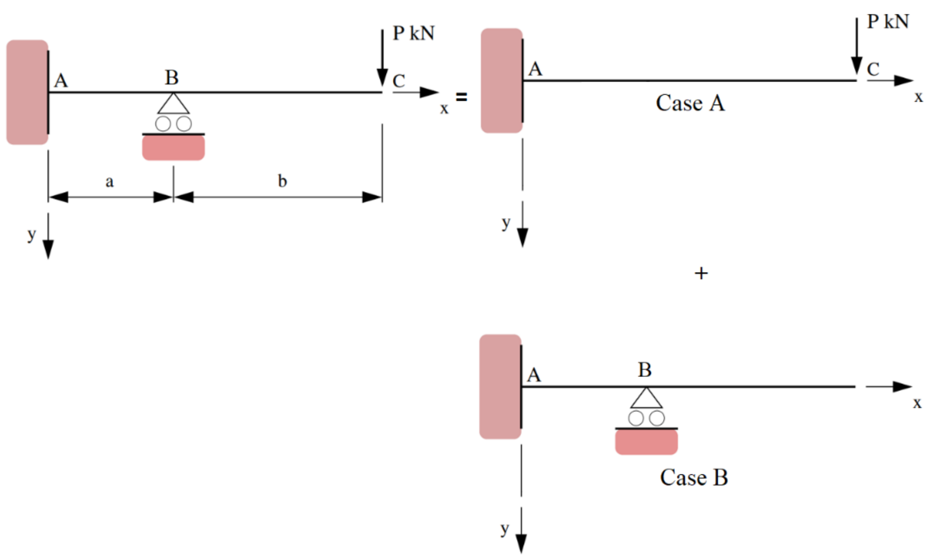


Figure 4: Front elevation view of a cantilever supported by a roller at point B.

[2c] Show that the vertical support reaction at B is:

$$V_b = \frac{P}{2}\left[\frac{3b + 2a}{a}\right]. \tag{5}$$

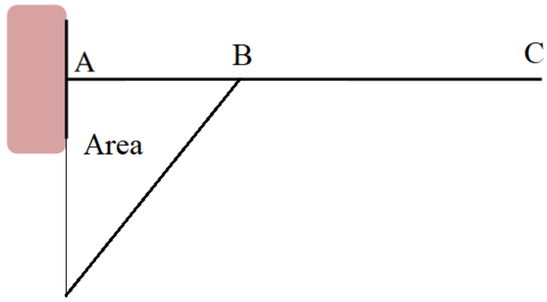
Using Superposition:



Case A:

$$\Delta_B = \frac{Pa^2}{6EI}(2a + 3b)\,; downward$$

Case B:



$$\Delta_B = Area * \bar{x}_{1B} = \left(\frac{1}{2} * a * \frac{V_B a}{EI}\right) * \left(\frac{2}{3} * a\right) = \frac{V_B a^3}{3EI}\,; upward$$

$$\Delta_{B_{total}} = 0 \rightarrow \frac{Pa^2}{6EI}(2a + 3b) = \frac{V_B a^3}{3EI} \rightarrow V_B = \frac{P}{2}\left(\frac{3b + 2a}{a}\right)\,; upward$$

[2d] Hence, derive a simple expression for the bending moment at A.

$$\sum M_A = 0 \rightarrow M_A + V_B * a - P * L = 0 \rightarrow M_A = -\frac{Pb}{2}\ (CCW), M_A = \frac{Pb}{2}\ (CW)$$

Finally, let’s replace the roller support below point B with a spring.

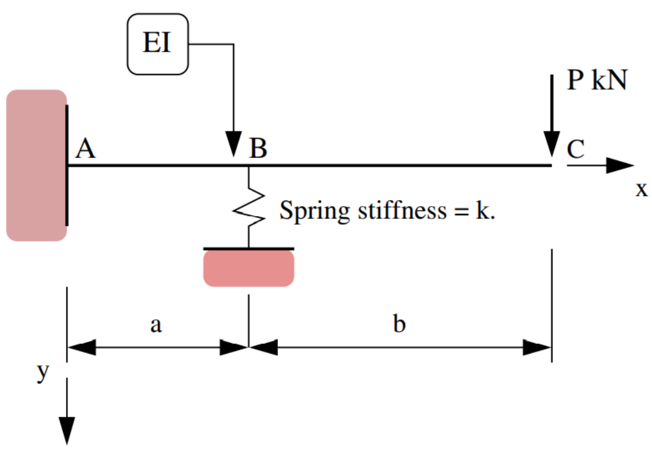


Figure 5: Cantilever supported by a spring at point B.

[2e] Show that the support reaction, V_b, is now given by the equation:

$$V_b\left[\frac{1}{k} + \frac{a^3}{3EI}\right] = \frac{Pa^2}{6EI}[3b + 2a]. \tag{6}$$

$$Recall: \ \Delta_{B_{total}} = \frac{Pa^2}{6EI}(2a+3b) - \frac{V_B a^3}{3EI}$$

$$There\ is\ a\ spring\ at\ point\ B: \ V_B = k\Delta_{B_{total}}$$

$$= k\left[\frac{Pa^2}{6EI}(2a+3b) - \frac{V_B a^3}{3EI}\right] \xrightarrow{simplifying} V_B\left(\frac{1}{k} + \frac{a^3}{6EI}\right) = \frac{Pa^2}{6EI}(3b+2a)$$

[2f] Explain why V_b for spring support (i.e., equation 6) is always lower than for roller support (i.e., equation 5).

Considering the case of spring support:

$$V_B = \frac{\frac{Pa^2}{6EI}(3b+2a)}{\frac{1}{k} + \frac{a^3}{6EI}} \xrightarrow{k\to\infty} V_B = \frac{\frac{Pa^2}{6EI}(3b+2a)}{\frac{a^3}{6EI}}$$

A roller support can be modeled as a spring with $k \to \infty$, therefore as shown above, V_B increases as the denominator decreases. Thus, V_B is always larger when there is a roller support compared to the case with a spring.

Problem 3: Consider the cantilevered beam structure shown in Figure 6.

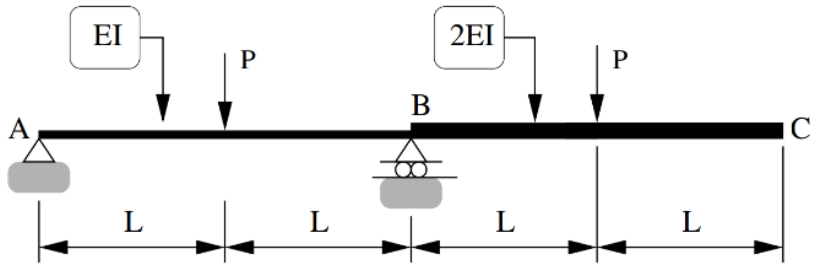
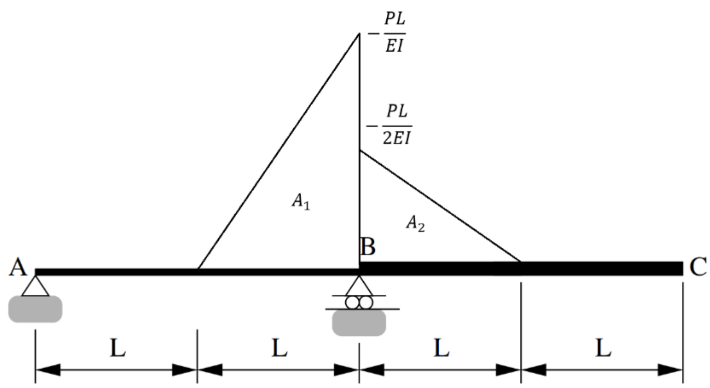


Figure 6: Front elevation view of a cantilevered beam structure.

Notice that segments A-B and B-C have cross-sectional properties EI and 2EI, respectively.

[3a] Use the method of moment-area to compute the rotation at point A.

$$\sum M_A = 0 \rightarrow V_B = 2P \ (upward), \sum F_y = 0 \rightarrow V_A = 0$$



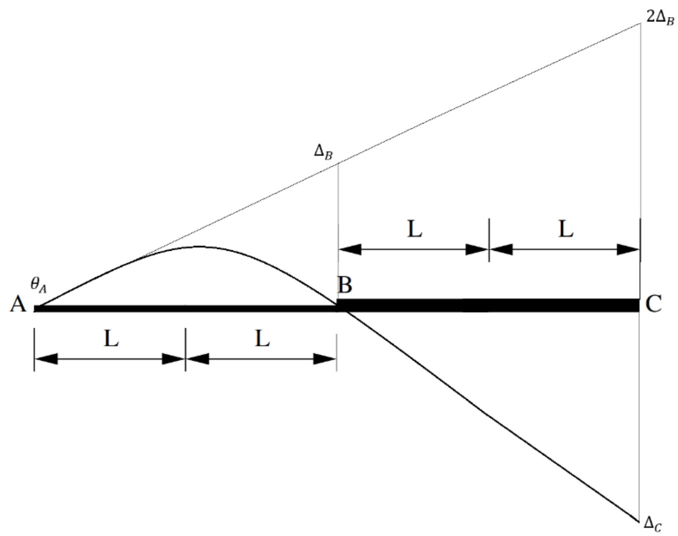
$$\begin{cases} A_1 = \frac{1}{2} * L * \left(-\frac{PL}{EI}\right) = -\frac{PL^2}{2EI} \\ A_2 = \frac{1}{2} * L * \left(-\frac{PL}{2EI}\right) = -\frac{PL^2}{4EI} \end{cases} \qquad \begin{cases} \bar{x}_{1c} = 2L + \frac{L}{3} = \frac{7}{3}L \\ \bar{x}_{2c} = L + \frac{2L}{3} = \frac{5}{3}L \end{cases}$$

$$\Delta_B = A_1 * \bar{x}_{1B} = \left(-\frac{PL^2}{2EI}\right) * \left(\frac{L}{3}\right) = -\frac{PL^3}{6EI}$$

$$\theta_A * 2L = \Delta_B \rightarrow \theta_A = -\frac{PL^2}{12EI} \rightarrow \theta_A = \frac{PL^2}{12EI} \ (CCW)$$

[3b] Use the method of moment-area to compute the vertical deflection of the beam at point C.

$$2\Delta_B + \Delta_C = A_1\bar{x}_{1c} + A_2\bar{x}_{2c} \rightarrow \Delta_C = -\frac{5}{4}\frac{PL^3}{EI} \ (downward)$$



[3c] Draw the deflected shape of the beam.