Homework 3 Solutions

(Due Date: April 7, 2023)

Problem 1: The three-pin parabolic arch shown in Figure 1 has a profile shape:

$$y(x) = \left[\frac{4f}{l^2}\right] x\,(l-x)$$

where f = 4m and L = 16m.

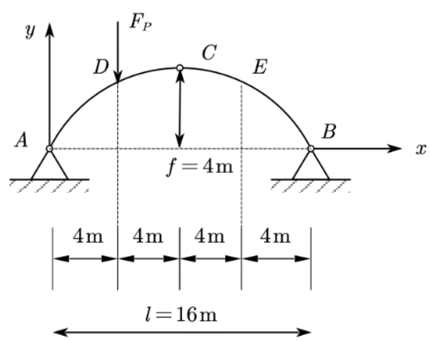
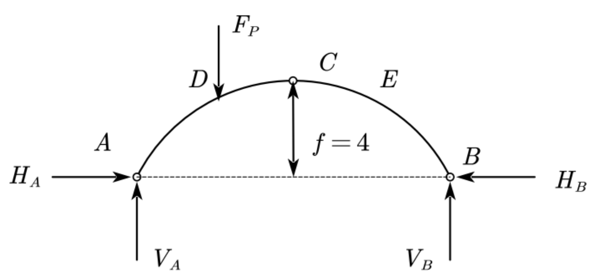


Figure 1: Elevation view of a parabolic three-pin arch.

Questions:

[1a] Calculate the horizontal and vertical components of reaction force at A and B.

(a) Draw the free body diagram of the arc structure:



$$\sum F_x = 0,\ H_A = H_B$$

$$\sum F_y = 0,\ V_A + V_B = F_p$$

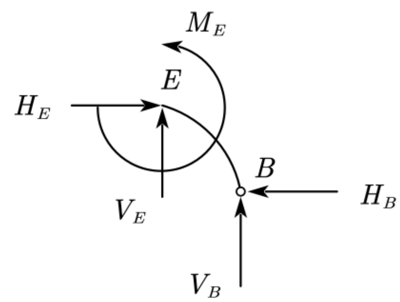
$$\sum M_A = 0,\ F_P \cdot 4 - V_B \cdot 16 = 0 \Rightarrow V_B = \frac{F_P}{4}(\uparrow) \Rightarrow V_A = \frac{3F_P}{4}(\uparrow)$$

Slice the arc structure at point C, and write the moment equilibrium about point C (left part) would get:

$$\sum M_C = 0,\ -F_P \cdot 4 + V_A \cdot 8 - H_A \cdot 4 = 0 \Rightarrow H_A = \frac{F_P}{2}(\rightarrow) \Rightarrow H_B = \frac{F_P}{2}(\leftarrow)$$

[1b] Calculate the internal forces (i.e., shear, moment and axial forces) at point E.

(b) Slice the arc structure at point E, and draw the free body diagram of the right part:

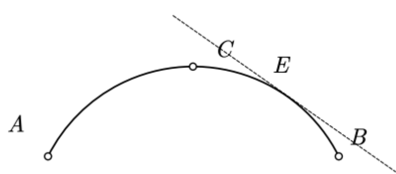


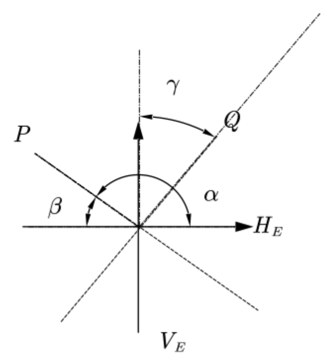
$$\sum F_x = 0,\ H_E = H_B \Rightarrow H_E = \frac{F_P}{2}\ (\rightarrow)$$

$$\sum F_y = 0,\ V_E + V_B = 0 \Rightarrow V_E = -\frac{F_P}{4}\ (\downarrow)$$

$$\sum M_E = 0,\ -M_E - V_B \cdot 4 + H_B \cdot y(12) = 0 \Rightarrow M_E = \frac{F_P}{2}\ (\curvearrowleft)$$

The axial force at point E is along with the tangent of the arc profile at point E and the shear force is perpendicular to the tangent of the arc profile. Thus, to calculate the axial and shear force, decomposition of H_E and V_E is needed:





$$\frac{dy}{dx}\Big|_{x=12} = \frac{4f}{l^2}(l - 2x)\Big|_{x=12} = -\frac{1}{2}$$

$$\tan\alpha = -\frac{1}{2} \Rightarrow \alpha = 153.43\,(\text{deg}) \Rightarrow \beta = 26.57\,(\text{deg}) \Rightarrow \gamma = \beta = 26.57\,(\text{deg})$$

$$\sin\beta = 0.447, \cos\beta = 0.894$$

$$P = V_E \sin\beta + H\cos\beta = \frac{F_P}{4} \cdot 0.447 + \frac{F_P}{2} \cdot 0.894 = 0.559 F_P\,(\text{compression})$$

$$Q = -V_E \cos\beta + H\sin\beta = -\frac{F_P}{4} \cdot 0.894 + \frac{F_P}{2} \cdot 0.447 = 0$$

[1c] Draw the bending moment diagram.

(c) Assume the moment at point x=x is positive in clockwise:

For section 0<=x<=4, write the moment equilibrium about point at x=x:

$$\sum M_{x=x} = 0,\ V_A \cdot x - H_A \cdot y(x) + M(x) = 0$$

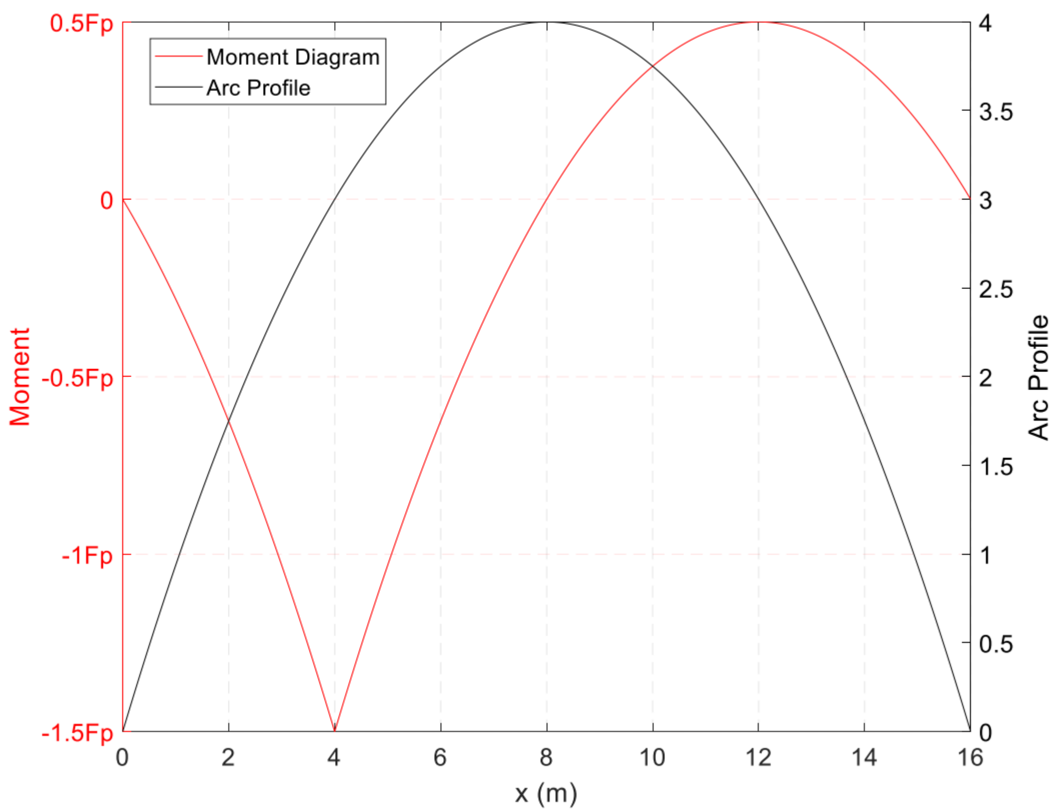
$$\Rightarrow M(x) = -\frac{3F_P}{4}x + \frac{F_P}{2}\left(\frac{4f}{l^2}x(l-x)\right) = -\frac{F_P}{32}x^2 - \frac{F_P}{4}x$$

For section 4<=x<=16,

$$\sum M_{x=x} = 0,\ V_A \cdot x - H_A \cdot y(x) - F_P \cdot (x-4) + M(x) = 0$$

$$\Rightarrow M(x) = -\frac{3F_P}{4}x + \frac{F_P}{2}\left(\frac{4f}{l^2}x(l-x)\right) = -\frac{F_P}{32}x^2 + \frac{3F_P}{4}x - 4F_P$$

Draw the moment diagram (Assume the value of F_p >=0):



Problem 2: Figure 2 shows an elevation view of a pre-fabricated steel building frame that is subject to a variety of snow and wind loadings.

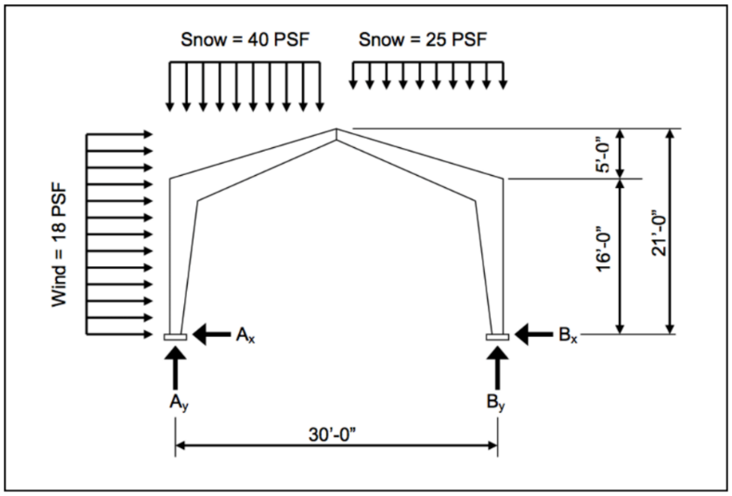
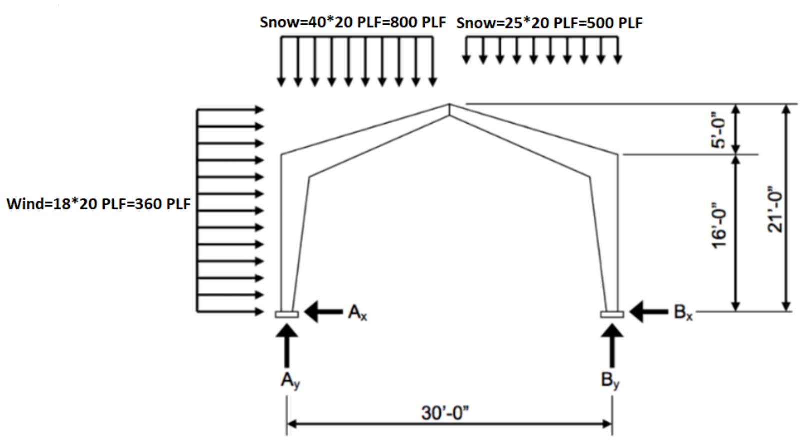
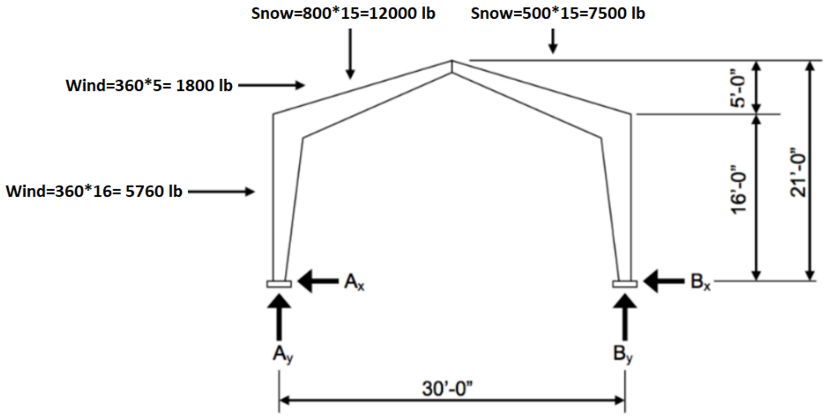


Figure 2: Elevation view of pre-fabricated steel building frame subject to snow and wind loadings.

Assuming that the frames are spaced at 20 ft centers, and that the foundation-level supports and roof apex are pinned (i.e., the frame can be modeled as a three-pinned arch), compute the vertical and horizontal reactions at the base supports.





Entire Arch:

$$\sum M_A = 0 \rightarrow 5760 * 8 + 1800 * 18.5 + 12000 * 7.5 + 7500 * 22.5 - B_y * 30 = 0 \rightarrow B_y = 11271\ lb$$

$$\sum F_y = 0 \rightarrow A_y - 12000 - 7500 + B_y = 0 \rightarrow A_y = 8229\ lb$$

Arch Segment BC:

Note: Point C is at the center of the arch (middle pin).

$$\sum M_C = 0 \rightarrow 7500 * 7.5 + B_x * 21 - B_y * 15 = 0 \rightarrow B_x = 5372.1\ lb$$

Entire Arch:

$$\sum F_x = 0 \rightarrow A_x - 5760 - 1800 + B_x = 0 \rightarrow A_x = 2187.9\ lb$$

Problem 3:

Analysis of a Three-Pinned Parabolic Arch. This question is inspired by the St. Louis Gateway Arch

shown on the class web page. We will compute the vertical and horizontal support reactions due to self-weight the arch alone, and explore the validity of approximations in the analysis along the way.

Since the mathematical details of this problem are a bit complicated, I suggest that you use Wolfram Alpha (see: https://www.wolframalpha.com) for the integration, and read the web page output carefully for hints on suitable simplifications.

Problem Setup. Figure 3 is a front elevation view of a three-pinned parabolic arch that has a profile:

$y(x) = kx^2$.

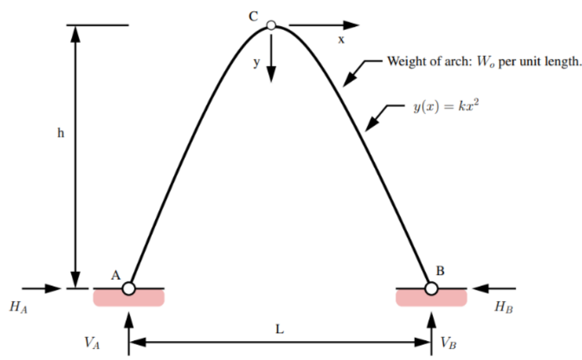


Figure 3: Front elevation view of a three-pinned parabolic arch.

The arch has height h, span L, and has self-weight W_o (N/m) along its profile. Points A, B and C are pins. Show all of your working.

[3a] Starting from first principles of geometry, show that the equivalent loading measured in the horizontal direction is:

$$w(x) = W_o\left[1 + 4k^2x^2\right]^{1/2}.$$

$$\sum F_y = 0 \rightarrow \omega(x)dx = \omega_0 ds \rightarrow \omega(x) = \omega_0 \frac{ds}{dx}$$

$$ds^2 = dx^2 + dy^2 \rightarrow (\frac{ds}{dx})^2 = 1 + (\frac{dy}{dx})^2$$

$$\omega(x) = \omega_0 \sqrt{1 + (\frac{dy}{dx})^2} = \omega_0\sqrt{1 + (2kx)^2} = \omega_0\sqrt{1 + 4k^2x^2}$$

[3b] Show that an approximate value of V_A is:

$$V_A \approx \frac{W_o L}{2}\left[1 + \frac{8}{3}\left(\frac{h}{L}\right)^2\right].$$

Notice that when h/L = 0, the arch becomes a straight beam and $V_A = W_o L/2$.

$$V_A = V_{\boldsymbol{B}} = \int_0^{L/2} \omega(x)dx$$

$$\int (1 + a^2x^2)^{\frac{1}{2}} = \frac{ax\sqrt{1 + a^2x^2} + \sin\mathrm{h}^{-1}(ax)}{2a} = \frac{ax\sqrt{1 + a^2x^2} + \log\left(ax + \sqrt{1 + a^2x^2}\right)}{2a}$$

$$\int (1 + a^2x^2)^{\frac{1}{2}} = x + \frac{a^2x^3}{6}; a = 2k, k = \frac{4h}{L^2}$$

$$V_A = \int_0^{L/2} \omega(x)dx = \omega_0(x + \frac{4k^2x^3}{6}\bigg|_0^{\frac{L}{2}}) = \omega_0(\frac{L}{2} + \frac{4k^2L^3}{6 \times 8}) = \frac{\omega_0 L}{2}[1 + \frac{8}{3}(\frac{h}{L})^2]$$

[3c] Using Wolfram Alpha, or otherwise, derive a formula for the moments about C due to self-weight of the arch alone, i.e.:

$$\int_0^{L/2} w(x)xdx.$$

All reasonable answers will be accepted.

$$\int_0^{\frac{L}{2}} \omega(x)xdx = \int_0^{\frac{L}{2}} \omega_0\sqrt{1 + 4k^2x^2}\, xdx = \frac{\omega_0(1 + 4k^2x^2)^{\frac{3}{2}}}{12k^2}\bigg|_0^{\frac{L}{2}} = \frac{\omega_0}{12k^2}(1 + 6k^2x^2)\bigg|_0^{\frac{L}{2}}$$

$$= \frac{\omega_0 L^2}{8}[1 + \frac{1}{24}\left(\frac{L}{h}\right)^2]$$

[3d] With equations 3 and 4 in place, write down and label the equation you would solve to compute the horizontal reaction force at A.

$$\sum M_c = 0 \rightarrow \int_0^{\frac{L}{2}} \omega(x)xdx = V_A\left(\frac{L}{2}\right) - H_A(h) \rightarrow H_A = \frac{V_A\left(\frac{L}{2}\right) - \int_0^{\frac{L}{2}} \omega(x)xdx}{h}$$

[3e] Now suppose that equation 3 is applied to the St. Louis Gateway Arch profile (see pic on class web page), where h/L = 1.

Does the computed value for V_A seem reasonable to you, or not? And if not, how you would correct the analysis? Either way, justify your answer.

$$\frac{h}{L} = 1 \xrightarrow{Approximate\ Value} V_A = \frac{\omega_0 L}{2}\left[1 + \frac{8}{3}\left(\frac{h}{L}\right)^2\right] = \frac{11}{3}\left(\frac{\omega_0 L}{2}\right) = \omega_0(\frac{11}{6}L)$$

$$For\ a\ Parabola: V_A = \int_0^{\frac{L}{2}} \omega(x)dx = \omega_0(Length\ of\ arc)$$

$$\sqrt{L^2 + \left(\frac{L}{2}\right)^2} = \frac{\sqrt{5}}{2}L < Length\ of\ arc < \frac{3}{2}L$$

$$Therefore, the\ approximate\ value\ of\ V_A\ is\ not\ reasonable\ \left(\frac{11}{6}L > \frac{3}{2}L\right).$$

$$\int (1 + a^2x^2)^{\frac{1}{2}} = \frac{ax\sqrt{1 + a^2x^2} + \log\left(ax + \sqrt{1 + a^2x^2}\right)}{2a};\ a = 2k, k = \frac{4h}{L^2}$$

$$V_A = \omega_0 \int_0^{\frac{L}{2}} (1 + a^2x^2)^{\frac{1}{2}} dx = \omega_0 \int_0^{\frac{L}{2}} (1 + 4k^2x^2)^{\frac{1}{2}} dx = \omega_0 (1.21L) \rightarrow \frac{\sqrt{5}}{2}L < 1.21L < \frac{3}{2}L$$

$$\rightarrow This\ is\ a\ better\ approximation.$$

Problem 4: The cable structure shown in Figure 4 carries a uniform load w_o (N/m) along its entire length.

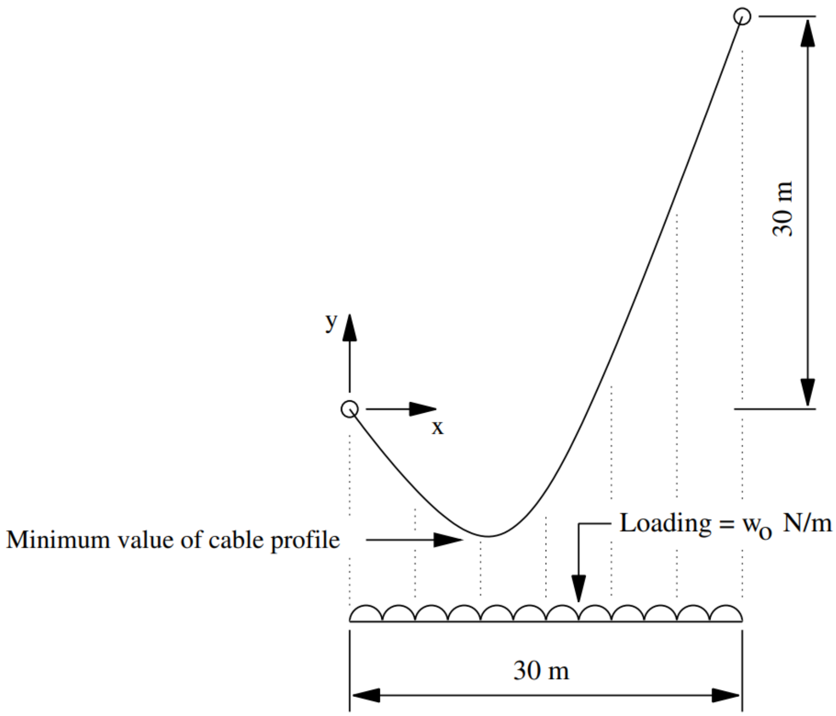


Figure 4: Elevation view of a pedestrian swing bridge.

[4a] Starting from first principles (i.e., the differential equation), show that cable profile is given by the equation:

$$y(x) = \frac{w_o x^2}{2H} + \left(1 - \frac{15 w_o}{H}\right) x.$$

$$\frac{d^2y}{dx^2} = \frac{w(x)}{H} = \frac{w_0}{H} \rightarrow \frac{dy}{dx} = \frac{w_0}{H} x + A \rightarrow y(x) = \frac{w_0}{2H} x^2 + Ax + B$$

$$\begin{cases} y(x=0) = 0 \\ y(x=30) = 30 \end{cases} \rightarrow \begin{cases} B = 0 \\ A = 1 - \dfrac{15 w_0}{H} \end{cases} \rightarrow y(x) = \frac{w_0 x^2}{2H} + \left(1 - \frac{15 w_0}{H}\right) x$$

Now let us assume that the minimum value of the cable profile occurs at x = 10.

[4b] Show that the horizontal cable force is:

$$H = 5 w_o.$$

$$\frac{dy}{dx} = \frac{w_0}{H} x + 1 - \frac{15 w_0}{H} \rightarrow \left.\frac{dy}{dx}\right|_{x=10} = 0 \rightarrow H = 5 w_0$$

[4c] Derive a simple expression for the maximum tensile force in the cable.

Note: Point A is the left end, and point B is the right end.

$$\sum M_A = 0 \rightarrow V_B * 30 - w_0 * 30 * \left(\frac{30}{2}\right) - H * 30 = 0 \rightarrow V_B = 20w_0$$

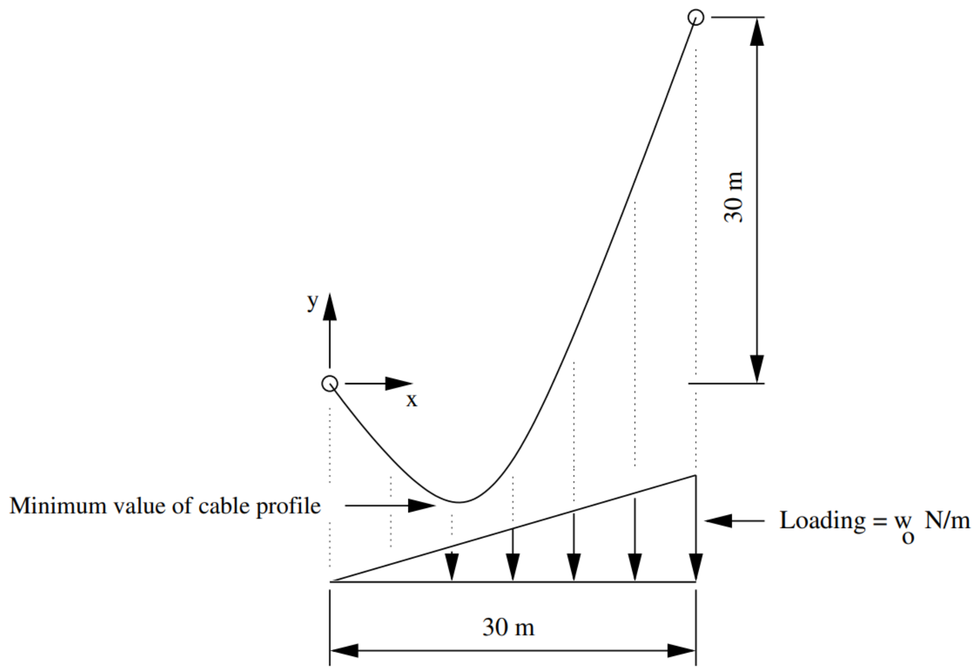
$$\sum F_y = 0 \rightarrow V_A + V_B = 30w_0 \rightarrow V_A = 10w_0$$

$$T_A = \sqrt{H^2 + {V_A}^2} = \sqrt{(5w_0)^2 + (10w_0)^2} = 11.1803w_0$$

$$T_B = \sqrt{H^2 + {V_B}^2} = \sqrt{(5w_0)^2 + (20w_0)^2} = 20.6155w_0$$

$$T(x) = T_A + \frac{T_B - T_A}{30}x = 11.1803w_0 + 0.3145w_0x$$

Problem 5: The cable structure shown in Figure below carries a triangular load that is zero at the left-hand support and increases to w_o (N/m) at the right-hand support.



[5a] Starting from first principles (i.e., the differential equation), show that cable profile is given by the equation:

$$y(x) = \frac{w_o x^3}{180H} + \left(1 - \frac{5w_o}{H}\right) x.$$

$$\frac{d^2y}{dx^2} = \frac{w(x)}{H} = \frac{w_0 x}{30H} \rightarrow \frac{dy}{dx} = \frac{w_0}{60H}x^2 + A \rightarrow y(x) = \frac{w_0}{180H}x^3 + Ax + B$$

$$\begin{cases} y(x=0) = 0 \\ y(x=30) = 30 \end{cases} \rightarrow \begin{cases} B = 0 \\ A = 1 - \dfrac{5w_0}{H} \end{cases} \rightarrow y(x) = \frac{w_0 x^3}{180H} + \left(1 - \frac{5w_0}{H}\right) x$$

Now let us assume that the minimum value of the cable profile occurs at x = 10.

[5b] Show that the horizontal cable force is:

$$H = \frac{20w_o}{6}.$$

$$\frac{dy}{dx} = \frac{w_0}{60H}x^2 + 1 - \frac{5w_0}{H} \rightarrow \left.\frac{dy}{dx}\right|_{x=10} = 0 \rightarrow H = \frac{10}{3}w_0$$

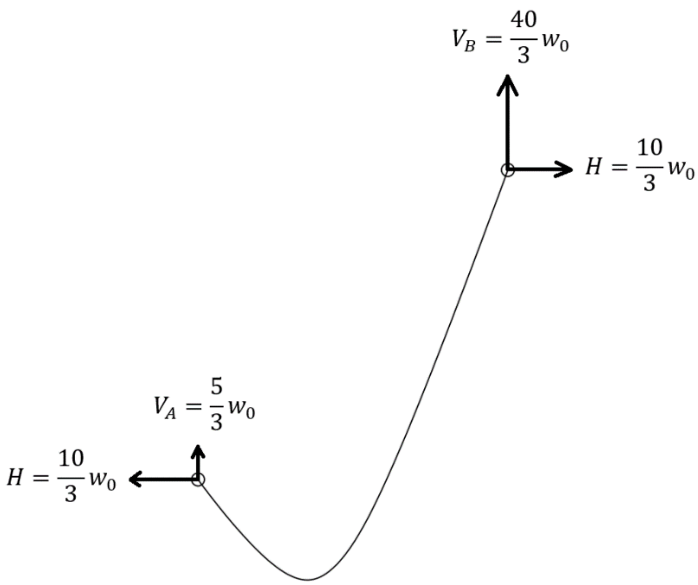
[5c] Draw and label a diagram showing the horizontal and vertical components of reaction force at the left and right-hand cable supports.

Note: Point A is the left end, and point B is the right end.

Method 1:

$$\sum M_A = 0 \rightarrow V_B * 30 - \frac{1}{2} * w_0 * 30 * \left(\frac{2}{3}\right) * 30 - H * 30 = 0 \rightarrow V_B = \frac{40}{3} w_0$$

$$\sum F_y = 0 \rightarrow V_A + V_B = \frac{1}{2} * w_0 * 30 = 15 w_0 \rightarrow V_A = \frac{5}{3} w_0$$



Method 2:

$$V = H \frac{dy}{dx} \rightarrow V_A = H \frac{dy}{dx}\bigg|_{x=0} = H\left(1 - \frac{5w_0}{H}\right) = H - 5w_0 = -\frac{5}{3} w_0$$

$$V = H \frac{dy}{dx} \rightarrow V_B = H \frac{dy}{dx}\bigg|_{x=30} = H\left(\frac{w_0}{60H} * 30^2 + 1 - \frac{5w_0}{H}\right) = \frac{40}{3} w_0$$