

ENCE 353 Midterm 2, Open Notes and Open Book

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Exam Format and Grading. This take home midterm exam is open notes and open book. You need to comply with the university regulations for academic integrity.

There are three questions. Partial credit will be given for partially correct answers, so please show all your working.

Please see the **class web page** for instructions on how to submit your exam paper.

Question	Points	Score
1	10	
2	10	
3	20	
Total	40	

Question 1: 10 points

Analysis of a Cantilever Beam with Moment Area. Consider the cantilever shown in Figure 1.

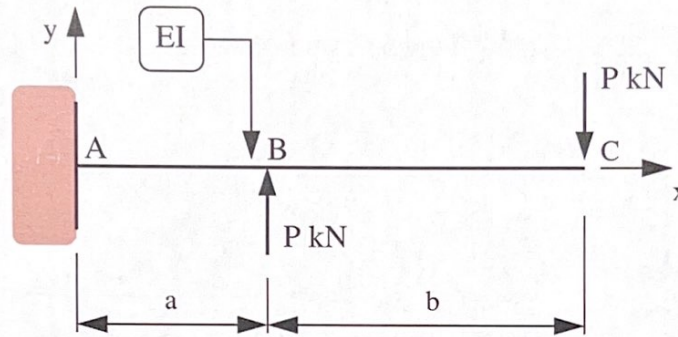
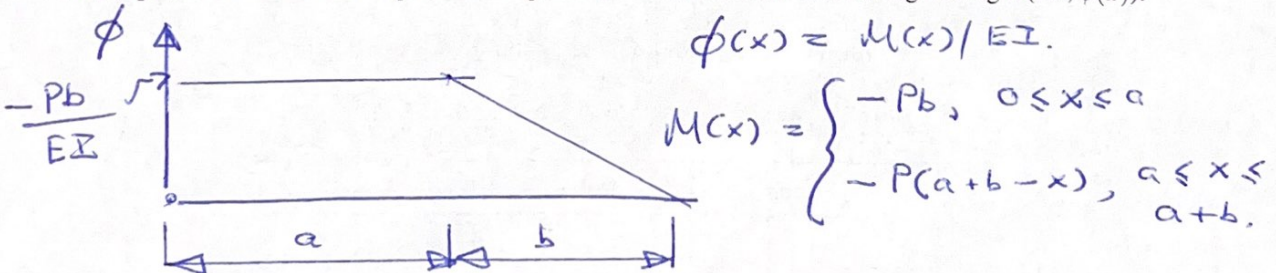


Figure 1: Front elevation view of a cantilever.

The cantilever has constant section properties, EI , along its entire length ($a+b$). Vertical loads of P (kN) are applied at points B and C. Notice that the y axis is pointing upwards – hence, if we apply the right-hand rule, positive rotations will be anti-clockwise.

[1a] (2 pts) Draw and label a diagram showing how the beam curvature varies along its length (i.e., $\phi(x)$).

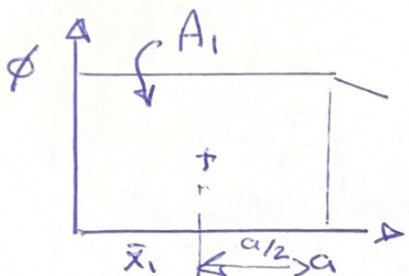


[1b] (2 pts) Use the method of moment-area to show that the rotation of point B is:

$$\theta(a) = \left[\frac{-Pba}{EI} \right]. \quad (1)$$

$$\begin{aligned} \theta(a) &= \text{area of } M/EI \text{ diagram between } x=0 \text{ \& } x=a. \\ &= -\frac{Pba}{EI}. \end{aligned}$$

[1c] (3 pts) Use the method of moment-area to show that the vertical displacement of point B is:



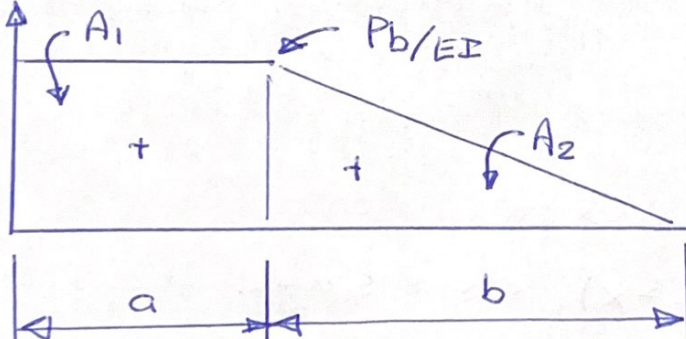
$$y(a) = \left[\frac{-Pba^2}{2EI} \right]. \quad (2)$$

$$A_1 = \frac{-Pba}{EI} \left. \begin{array}{l} y(a) = A_1 \cdot \bar{x}_1 \\ \bar{x}_1 = a/2. \end{array} \right\} = -\frac{Pba^2}{2EI}.$$

[1d] (3 pts) Use the method of moment-area to show that the vertical displacement of point C is:

$$y(a+b) = \left[\frac{-Pb}{6EI} \right] [6ab + 3a^2 + 2b^2]. \quad (3)$$

Note: Notice that when $a = 0$, equation 3 simplifies to the formula we have seen many times in class.



$$A_1 = \frac{Pba}{EI}, \quad A_2 = \frac{1}{2} \frac{Pb^2}{EI},$$

$$\bar{x}_1 = b + \frac{a}{2}, \quad \bar{x}_2 = \frac{2}{3}b.$$

$$y(a+b) = - [A_1 \bar{x}_1 + A_2 \bar{x}_2].$$

$$= - \frac{Pba}{EI} \left[b + \frac{a}{2} \right] - \frac{Pb^2}{2EI} \left[\frac{2}{3}b \right].$$

$$= - \frac{Pb}{6EI} [6ab + 3a^2 + 2b^2].$$

Question 2: 10 points

Elastic Curve for a Cantilever Beam Structure. Consider the cantilever shown in Figure 2.

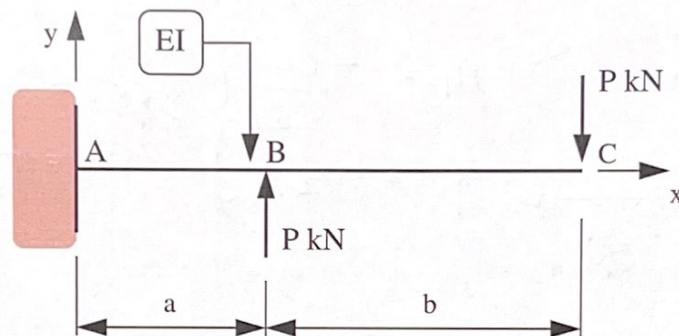


Figure 2: Front elevation view of a cantilever.

The cantilever has constant section properties, EI , along its entire length $(a+b)$. Vertical loads of P (kN) are applied at points B and C.

[2a] (2 pts) Write a mathematical formula, $M(x)$, for the bending moment along the beam as a function of x .

$$M(x) = \begin{cases} -Pb, & 0 \leq x \leq a \\ -P(a+b-x), & a \leq x \leq a+b. \end{cases}$$

[2b] (3 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{M(x)}{EI} \right], \quad (4)$$

and appropriate boundary conditions, show that in the interval $0 \leq x \leq a$, the beam displacement and rotation are:

$$y(x) = \left[\frac{-Pbx^2}{2EI} \right] \quad \text{and} \quad \theta(x) = \left[\frac{-Pbx}{EI} \right]. \quad (5)$$

Show all of your working.

Within the interval $0 \leq x \leq a$,

$$\frac{d^2y}{dx^2} = \frac{-Pb}{EI}.$$

Integrating twice gives:

$$\frac{dy}{dx} = \frac{-Pbx}{EI} + A$$

$$y(x) = \frac{-Pbx^2}{2EI} + Ax + B.$$

Boundary conditions:

$$y'(0) = \left. \frac{dy}{dx} \right|_{x=0} = 0 \Rightarrow A = 0, B = 0.$$

Thus,

$$y(x) = \frac{-Pbx^2}{2EI} \quad \text{and} \quad \frac{dy}{dx} = \theta(x) = \frac{-Pbx}{EI}.$$

[2c] (5 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{M(x)}{EI} \right], \quad (6)$$

and appropriate boundary conditions, show that within the interval $a \leq x \leq a+b$, the beam displacement is:

$$y(x) = \left[\frac{-P}{6EI} \right] [3(a+b)x^2 - x^3 - 3a^2x + a^3]. \quad (7)$$

Hint: The algebra for this question is a bit tricky so I suggest that you work out the details on paper and submit a tidy summary of your procedure. Also notice that equation 7 is consistent with equations 2 and 3 in Question 1.

Boundary conditions:

$$\text{At } x=a, \quad y(a) = \frac{-Pba^2}{2EI} \quad \& \quad \frac{dy}{dx} = \frac{-Pba}{EI}.$$

$$\text{At } x=a+b, \quad \left. \frac{d^2y}{dx^2} \right|_{x=a+b} = 0 \quad \leftarrow \text{because } M(a+b) = 0$$

Within our interval of interest, $M(x) = -P(a+b-x)$

Hence,

$$EI \frac{d^2y}{dx^2} = -P(a+b-x). \quad \text{--- } \textcircled{A}$$

Integrating twice:

$$EI \frac{dy}{dx} = -P(a+b)x - \frac{Px^2}{2} + A. \quad \text{--- } \textcircled{B}$$

$$EI y(x) = \frac{-P}{2}(a+b)x^2 - \frac{Px^3}{6} + Ax + B. \quad \text{--- } \textcircled{C}$$

Question 2c continued:

At $x = a$,

$$-EI \frac{dy}{dx} = Pba = \underbrace{P(a+b)a}_{\text{equation (B)}} - \frac{Pa^2}{2} + A.$$

boundary condition

$$\Rightarrow A = -\frac{a^2}{2}.$$

$$-EI y(a) = \frac{Pba^2}{2} = \underbrace{\frac{P(a+b)a^2}{2} - \frac{Pa^3}{6} - \frac{Pa^3}{2}}_{\text{equation (C)}} + B.$$

boundary condition

$$\Rightarrow B = \frac{a^3}{6}.$$

Collecting terms:

$$y(x) = \frac{-P}{6EI} [3(a+b)x^2 - x^3 - 3a^2x + a^3]. \quad \text{--- (D)}$$

Note: Equation (D) is consistent with our results from moment area.

Question 3: 20 points

Moment-Area Method, Principle of Superposition, Compatibility of Displacements. Figure 3 is a front elevation view of a cantilevered beam carrying a single point load P . EI is constant along the beam structure A-B-C-D.

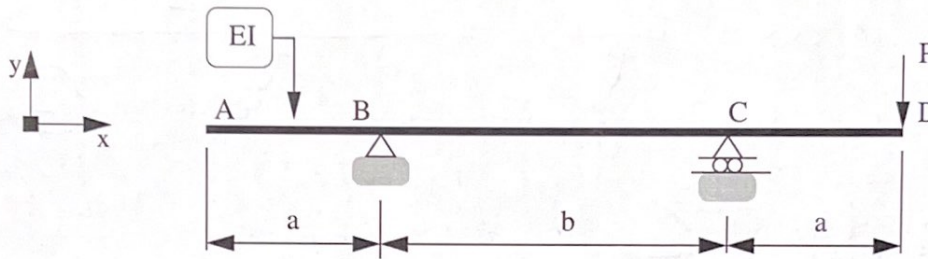
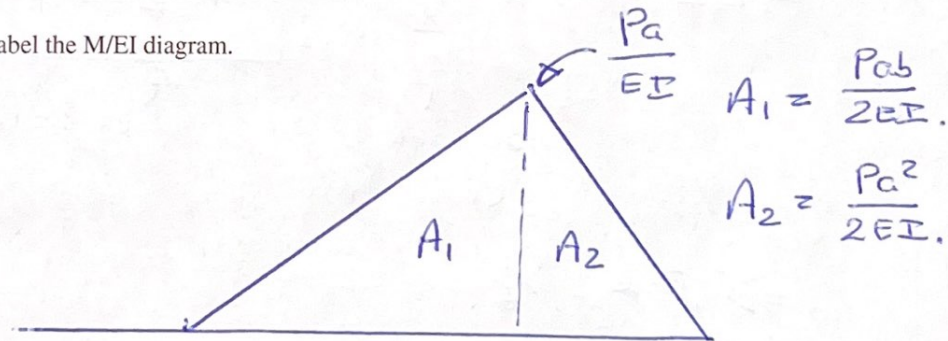
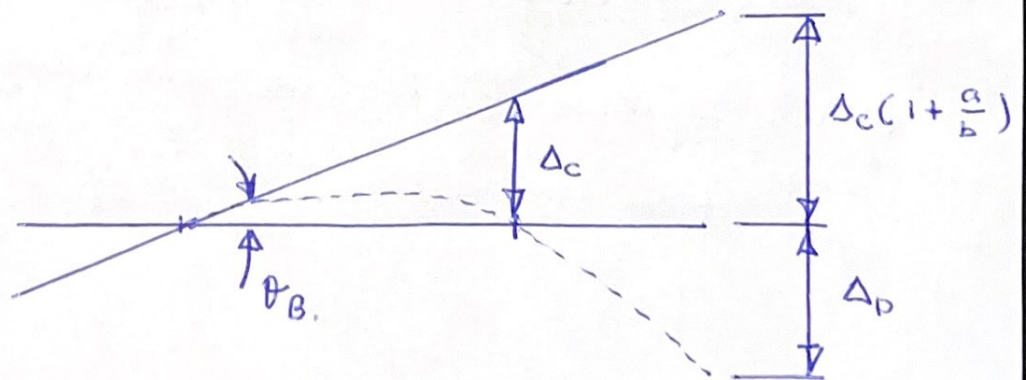


Figure 3: Cantilevered beam carrying a single applied load P .

[3a] (2 pts) Draw and label the M/EI diagram.



[3b] (2 pts) Draw and label the **moment area analysis diagram** (i.e., with rotations, tangents, displacements, etc) for this problem.



[3c] (3 pts) Use the method of **moment area** to show that the anticlockwise rotation of the beam at B is:

$$\theta_B = \frac{Pab}{6EI} \quad (8)$$

and the clockwise rotation of the beam at C is:

$$\theta_C = \frac{Pab}{3EI} \quad (9)$$

Show all of your working.

Moment Area. $\Delta_c = \left(\frac{b}{3}\right) A_1 = \frac{Pab^2}{6EI}$

$$\theta_B = \left(\frac{\Delta_c}{b}\right) = \frac{Pab}{6EI}$$

From the first theorem of moment area:

$$\theta_B + \theta_C = A_1 = \frac{Pab}{2EI}$$

↑ anticlockwise
↑ clockwise

$$\Rightarrow \theta_C = \frac{Pab}{2EI} - \frac{Pab}{6EI} = \frac{Pab}{3EI} \checkmark$$

[3d] (3 pts) Use the method of **moment area** to show that the vertical deflection of the beam at points A and D (measured downwards) is:

$$y_A = \frac{Pa^2b}{6EI} \quad (10)$$

and

$$y_D = \frac{Pa^2}{3EI}(a+b). \quad (11)$$

Show all of your working.

$$\text{Downwards deflection at A.} = \theta_B \cdot a = \frac{Pa^2b}{6EI}.$$

Use moment-area to find deflection at D.

$$\Delta_C = A_1 \left(\frac{b}{3} \right)$$

$$\Delta_C \left(1 + \frac{a}{b} \right) + \Delta_D = A_2 \left(\frac{2}{3}a \right) + A_1 \left(\frac{b}{3} + a \right)$$

$$\Rightarrow \Delta_D = \frac{Pa^2}{3EI}(a+b).$$

Now consider Figure 4, a front elevation view of a multi-span beam structure that carries a point load $2P$ (N) at the midspan of beam segment B-C. EI is constant along the beam structure A-B-C-D.

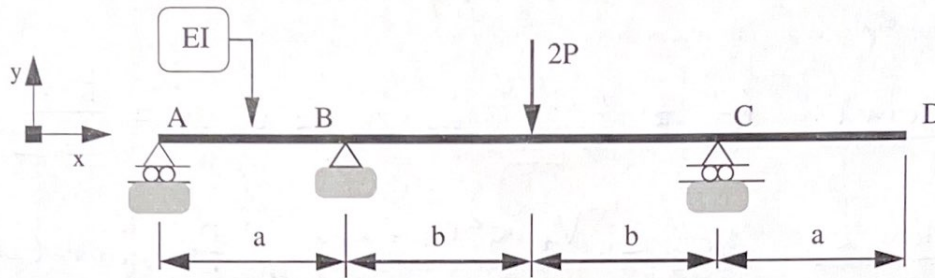


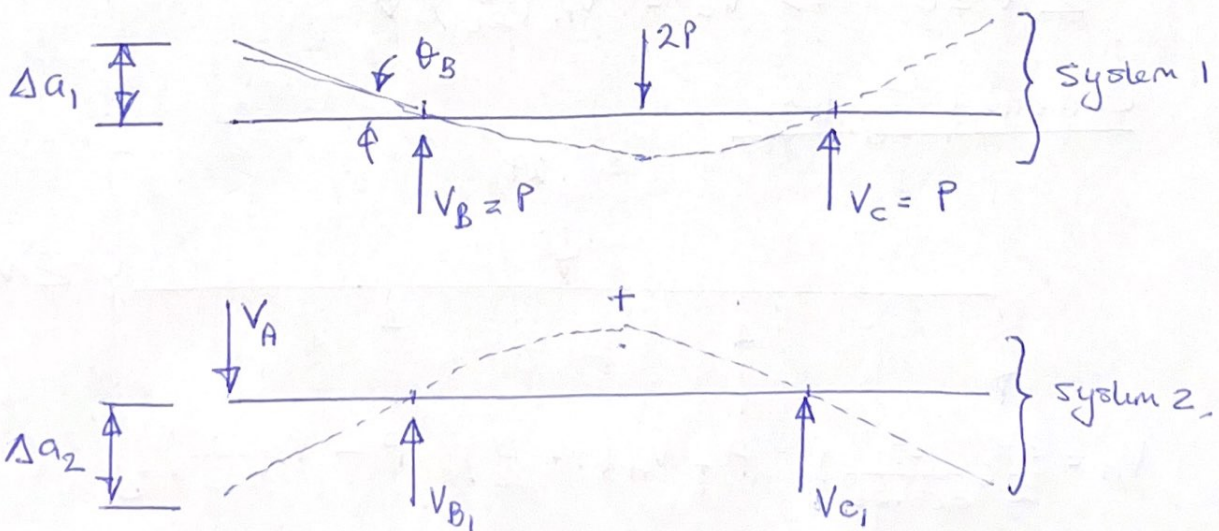
Figure 4: Multi-span beam carrying a point load on segment B-C.

[3e] (2 pts) Compute the degree of indeterminacy for this structure.

$\uparrow = 1$. ← statically indeterminate to degree 1.

[3f] (3 pts) Draw and label a diagram indicating how the **principle of superposition** can be used to simplify the analysis of the multi-span beam structure.

Here's one way -- release roller support at A. Then, apply superposition + displacement compatibility.



[3g] (5 pts) Using the method of moment area or otherwise, compute the support reactions at supports A, B and C.

Check that you reaction forces are in equilibrium with the applied loads.

System 1: $\theta_B = \frac{1}{2} \frac{Pb^2}{EI} \Rightarrow \Delta_{A_1} = \theta_B \cdot a = \frac{Pab^2}{2EI} \uparrow$

System 2: $\Delta_{A_2} = \frac{V_A \cdot a^2 (a+2b)}{3EI} \downarrow$ from part (3d)

Geometric compatibility: $\Delta_{A_1} \uparrow + \Delta_{A_2} \downarrow = 0$

$\Rightarrow V_A = \frac{3}{2} P \frac{b^2}{a(a+2b)} \downarrow$

$\sum M_C = 0 \Rightarrow V_A (a+2b) = V_{B_1} \cdot 2b$

$\Rightarrow V_{B_1} = V_A \left(1 + \frac{a}{2b}\right) \uparrow$

$\sum V = 0 \Rightarrow V_C = V_A \left(\frac{a}{2b}\right) \downarrow$

Combined System.

