Department of Civil and Environmental Engineering, Spring Semester, 2023

ENCE 353 Final Exam, Open Notes and Open Book

Name : ________________________

Exam Format and Grading. This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

Answer Question 1. Then answer **three of the five** remaining questions. Partial credit will be given for partially correct answers, so please show all your working.

IMPORTANT: Only the **first four questions** that you answer will be graded, so please **cross out the two questions you do not want graded** in the table below. Also, before submitting your exam, check that **every page has been scanned correctly**.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	50	

Question 1: 20 points

COMPULSORY: Moment-Area, Virtual Forces, Superposition. Figure 1 is a front elevation view of a simple beam structure carrying two external loads P. The beam has section properties EI along its entire length.

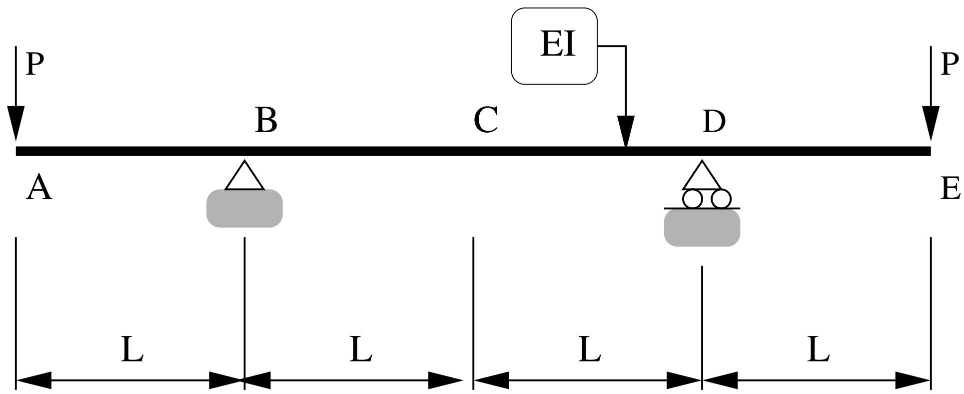

Figure 1: Simple beam structure will symmetric loads P.

[1a] (3 pts) Use the method of **moment area** to show that the beam rotation at D (measured clockwise) is:

$$\theta_D = \frac{PL^2}{EI}. \tag{1}$$

[1b] (3 pts) Use the method of **moment area** to show that the beam deflection at C (measured upwards) is:

$$\triangle_C = \frac{1}{2}\frac{PL^3}{EI}. \tag{2}$$

[1c] (3 pts) Use the method of **moment area** to show that the that the beam deflection at E (measured downwards) is:

$$\triangle_E = \frac{4}{3}\frac{PL^3}{EI}. \tag{3}$$

[1d] (4 pts) A function is said to be even if it has the property f(x) = f(-x) (i.e., it is symmetric about the y axis). And a function is said to be odd if it has the property g(x) = - g(-x) (i.e., it is skew-symmetric about the y axis). One example of an even function is cos(x), and one example of an odd function is sin(x).

Using high-school-level calculus (or otherwise), show that:

$$\int_{-h}^{h} f(x)g(x)dx = 0. \tag{4}$$

Please show all of your working.

Figure 2 shows the same beam structure, but now the external loads are rearranged so that one load points down and one load points up.

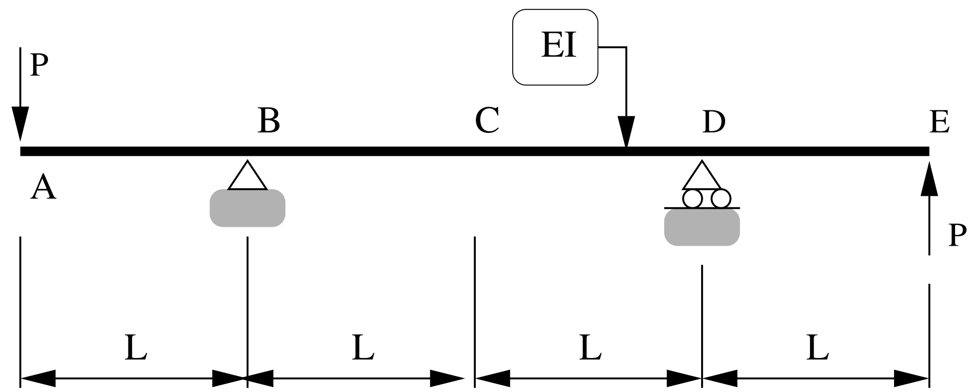

Figure 2: Simple beam structure with skew-symmetric loading pattern.

[1e] (4 pts) Use the method of **virtual forces** and a coordinate system positioned at C to show that the vertical displacement of C is zero, i.e., $\triangle_C = 0$.

Now consider the problem.

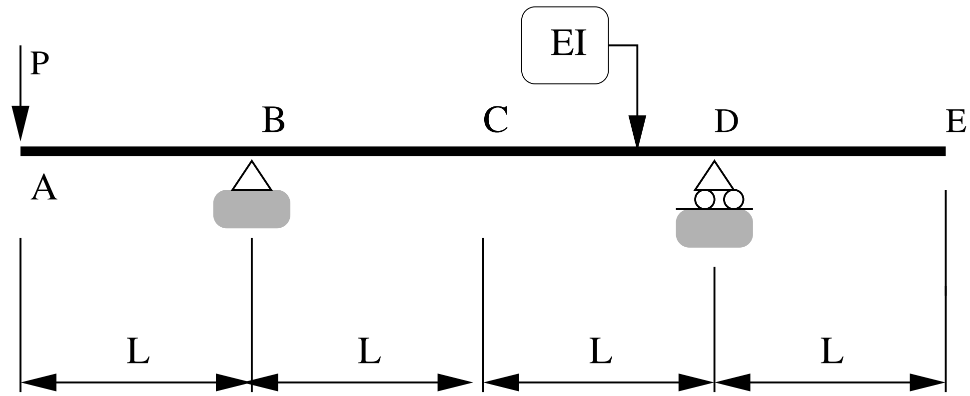

Figure 3: Simple beam structure with external load P.

[1f] (3 pts) Use your answers from parts [1b] and [1e] to write down an expression for the vertical deflection at C due to the loading pattern shown in Figure 3.

`Note`: You should find this is a one line answer.

Question 2: 10 points

OPTIONAL: Derive Elastic Curve for Beam Deflection. Figure 5 is a front elevation view of a cantilevered beam carrying a single point load P. EI is constant along the beam structure A-B-C-D.

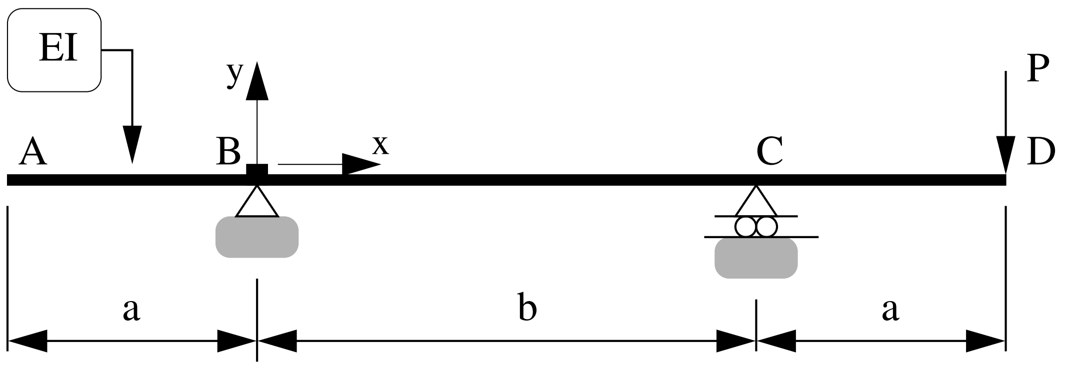

Figure 4: Cantilevered beam carrying a single applied load P.

[2a] (5 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{-M(x)}{EI}\right], \tag{5}$$

(notice the minus sign on M(x)) and appropriate boundary conditions, show that that rotation of the beam at C is:

$$\theta(b) = \frac{-Pab}{3EI}. \tag{6}$$

Now let's move the x-y coordinate system to point C, i.e.,

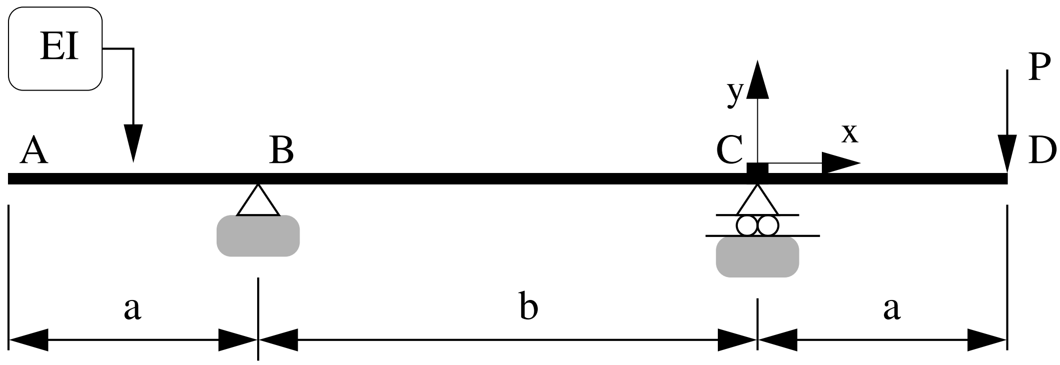

Figure 5: Cantilevered beam carrying a single applied load P.

[2b] (5 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{-M(x)}{EI} \right], \tag{7}$$

in the new coordinate system, show that that vertical deflection (measured downwards) at D is:

$$y_D = \frac{Pa^2}{3EI} (a+b). \tag{8}$$

Show all of your working.

Question 3: 10 points

OPTIONAL: Computing Displacements with the Method of Virtual Forces. Figure 6 is a front elevation view of a dog-leg cantilever beam carrying a point load P (N) at point D.

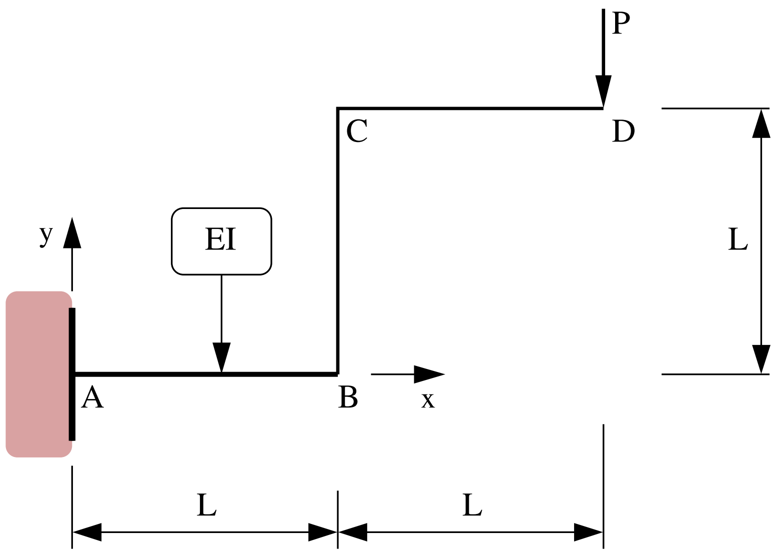

Figure 6: Dog-leg cantilever beam carrying end moment M (N.m).

The flexural stiffness EI is constant along A-B-C-D. The axial stiffness EA is very high and, as such, axial displacements can be ignored in the analysis.

[3a] (4 pts) Use the method of **virtual forces** to show that the clockwise rotation of the beam at point D is:

$$\theta_d = \frac{3PL^2}{EI}. \tag{9}$$

[3b] (6 pts) Use the method of **virtual forces** to show that the horizontal (measured left-to-right) and vertical (measured downwards) displacements at D are:

$$\triangle_h = \frac{2PL^3}{EI}. \tag{10}$$

and

$$\triangle_v = \frac{11}{3}\frac{PL^3}{EI}. \tag{11}$$

respectively. Show all of your working.

Question 3b continued ...

Question 4: 10 points

OPTIONAL: Bending Moment and Curvature in an Elastic Beam. Figure 7 is a front elevation view of a simply supported beam that carries a trapezoid load.

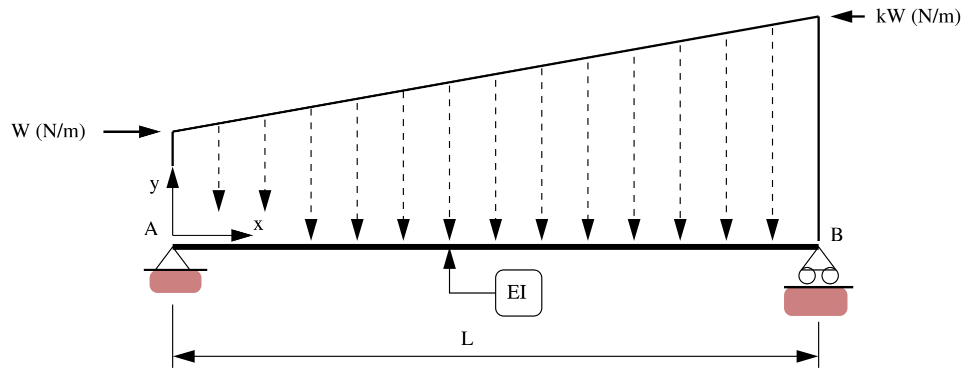

Figure 7: Simply supported beam carrying a trapazoid load.

The load increases from W (N/m) at x = 0, to kW (N/m) at x = L, where k is a non-negative constant. Thus, the total beam loading is $\frac{WL}{2}(1+k)$.

[4a] (2 pts). Starting from first principles of engineering, show that the vertical reactions at A and B are:

$$V_A = \frac{WL}{6}(2+k) \quad \text{and} \quad V_B = \frac{WL}{6}(1+2k)\,. \tag{12}$$

[4b] (3 pts). Show that the bending moment at point x is:

$$M(x) = \frac{WL^2}{6}\left(\frac{x}{L}\right)\left[(2+k) - 3\left(\frac{x}{L}\right) + (1-k)\left(\frac{x}{L}\right)^2\right]. \tag{13}$$

Notice that M(0) = M(L) = 0, regardless of the value of k.

The math for this part is a bit tedious – hence, I suggest you work out a solution on a separate sheet of paper, then write a tidy solution here.

[4c] (3 pts). Hence, show that the location of maximum curvature ϕ in the beam corresponds to the solution of the quadratic equation:

$$3\,(1-k)\,x^2 - 6Lx + (2+k)\,L^2 = 0. \tag{14}$$

[4d] (2 pts). For the case where k = 1 (i.e., a constant uniform loading), use equations 13 and 14 to determine the position and value of the maximum bending moment.

Question 5: 10 points

OPTIONAL: Use Principle of Virtual Work to Compute Displacements. Consider the articulated cantilever beam structure shown in Figure 8.

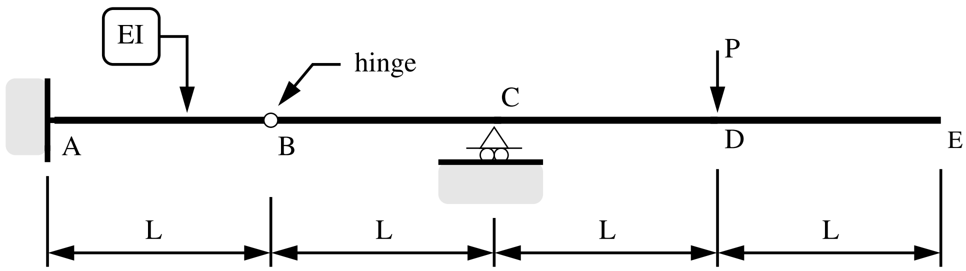

Figure 8: Elevation view of articulated cantilever beam structure.

At Point A, the cantilever is fully fixed (no movement) to a wall. Point B is a hinge. Both members have cross section properties EI. A single point load **P** (N) is applied at node D as shown in the figure.

[5a] (2 pts). Draw and label the bending moment diagram for this problem.

[5b] (2 pts). Qualitatively sketch the deflected shape. Indicate regions of tension/compression, and any points where slope of the beam is discontinuous.

[5c] (6 pts). Use the method of **virtual forces** to compute the **vertical displacement** and **end rotation** of the beam at E.

Show all of your working.

Question [5c] continued:

Question 6: 10 points

OPTIONAL: Zero-Force Memebers, Virtual Work, and Flexibility Matrices. Consider the truss structure shown in Figure 9.

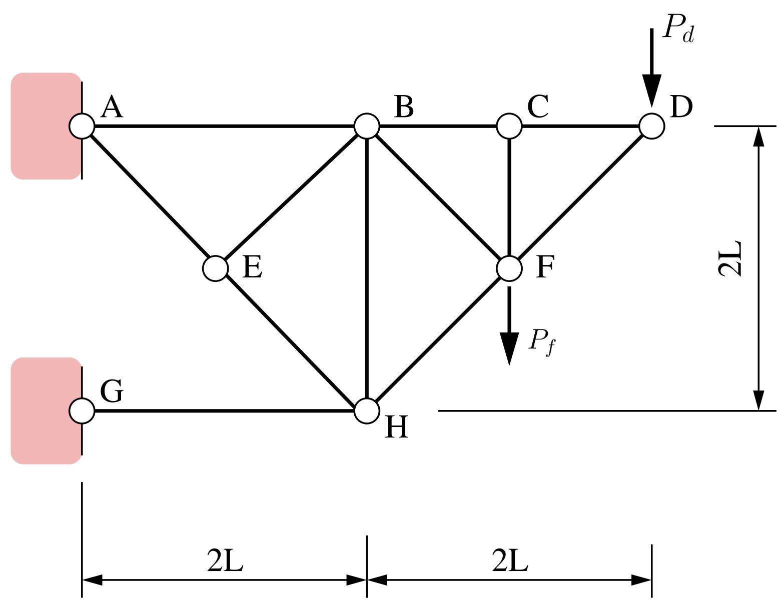

Figure 9: Elevation view of a pin-jointed truss.

The horizontal and vertical degrees of freedom are fully-fixed at supports A and G. The truss carries vertical loads P_d and P_f at nodes D and F, respectively. All frame members have cross section properties AE.

[6a] (3 pts) Use the method of **virtual forces** to compute the vertical deflection at node D due to load P_d alone (i.e., P_f = 0).

Question [6a] continued:

[6b] (3 pts) Use the method of **virtual forces** to compute the vertical deflection at node F due to load P_f alone (i.e., P_d = 0).

[6c] (4 pts) Use the method of **virtual forces** to compute the two-by-two flexibility matrix connecting the vertical displacements at points D and F to applied loads P_d and P_f, i.e., as a function of P_d, P_f, L and AE.

$$\begin{bmatrix} \triangle_d \\ \triangle_f \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_d \\ P_f \end{bmatrix}. \tag{15}$$