Department of Civil and Environmental Engineering, Fall Semester, 2022

ENCE 353 Final Exam, Open Notes and Open Book

Name : Austin.

Exam Format and Grading. This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

Answer Question 1. Then answer **three of the five** remaining questions. Partial credit will be given for partially correct answers, so please show all your working.

IMPORTANT: Only the **first four questions** that you answer will be graded, so please **cross out the two questions you do not want graded** in the table below. Also, before submitting your exam, check that **every page has been scanned correctly**.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	50	

Question 1: 20 points

COMPULSORY: Computing Displacements with the Method of Virtual Forces. Figure 1 is a front elevation view of a cantilevered beam carrying a single point load P. EI is constant along the beam structure A-B-C-D.

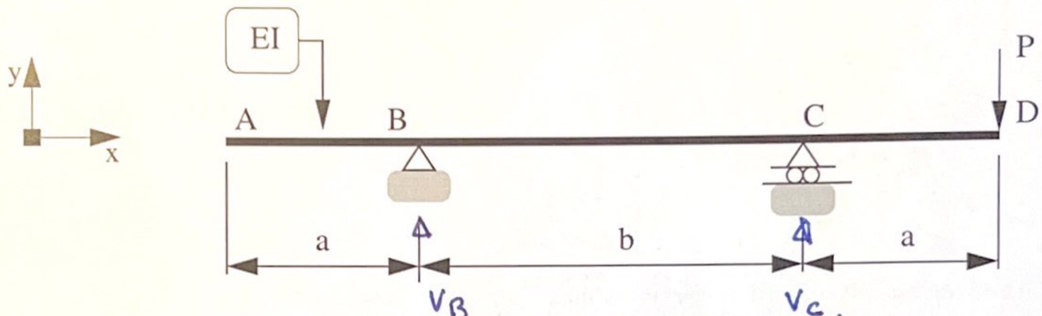
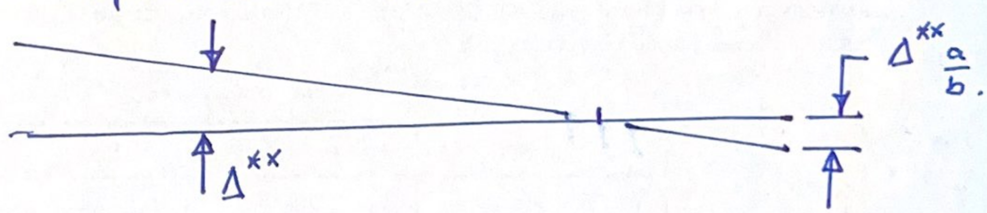


Figure 1: Cantilevered beam carrying a single applied load P.

[1a] (3 pts) Use the method of **virtual displacements** to compute the support reactions at B and C.

Apply unit displacement at B.

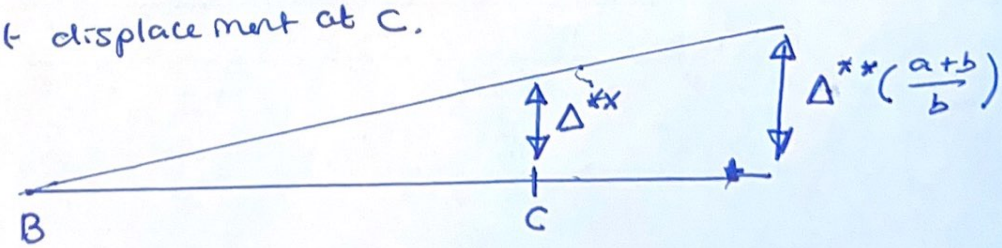


EWD = 0

$$\Rightarrow V_B \cdot \Delta^{**} + \left(\frac{a}{b}\right)\Delta^{**} P = 0$$

$$\Rightarrow V_B = -\left(\frac{a}{b}\right)P \quad \text{— ①}$$

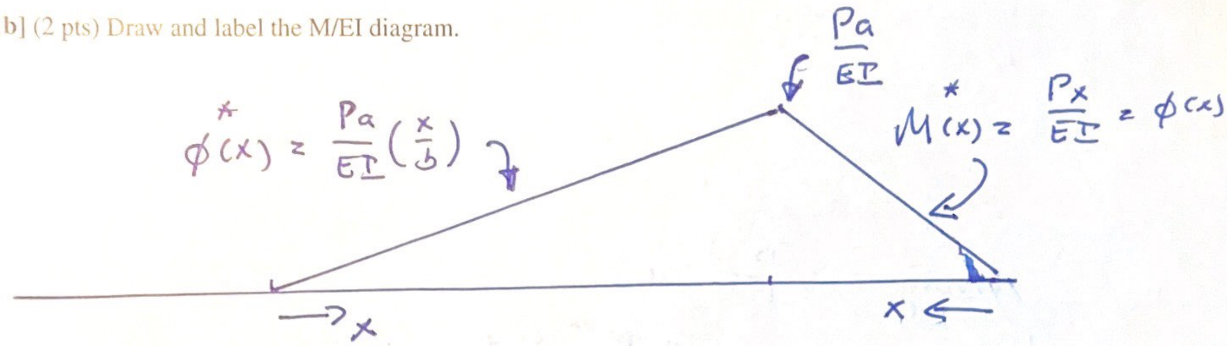
Apply unit displacement at C.



EWD = 0. $\quad V_C \cdot \Delta^{**} + \Delta^{**}\left(\frac{a+b}{b}\right)\cdot(-P) = 0$

$$\Rightarrow V_C = P\left(\frac{a+b}{b}\right). \quad \text{— ②}$$

[1b] (2 pts) Draw and label the M/EI diagram.



[1c] (5 pts) Use the method of **virtual forces** to show that the vertical deflection (measured downwards) at D is:

$$y_D = \frac{Pa^2}{3EI}(a+b). \tag{1}$$

Show all of your working.

$$\Delta_D = \underbrace{\int_0^b \left(\frac{Pa}{EI}\right)\left(\frac{x}{b}\right)\cdot\left(\frac{x}{b}\right) a \, dx}_{\frac{Pa^2 b}{3EI} = I_1} + \underbrace{\int_0^a \frac{P}{EI}\cdot x^2 dx}_{\frac{Pa^3}{3EI} = I_2}$$

$$\Delta_D = I_1 + I_2 = \frac{Pa^2}{3EI}(a+b).$$

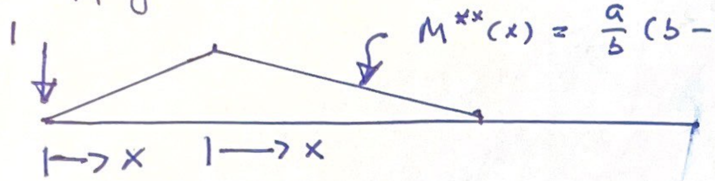
[1d] (5 pts) Use the method of **virtual forces** to show that the vertical deflection (measured downwards) at A is:

$$y_A = \frac{Pa^2b}{6EI} \tag{2}$$

Show all of your working.

Apply unit force at A.

$M^{**}(x) = \frac{a}{b}(b-x)$.



$M^{**}(x) = x$

$$\Delta_A = \int_0^b \underbrace{\frac{Pa}{EI}\cdot\left(\frac{x}{b}\right)}_{\phi(x)} \cdot \underbrace{\frac{a}{b}(b-x)}_{M^{**}(x)}\,dx = \frac{Pa^2b}{6EI}.$$

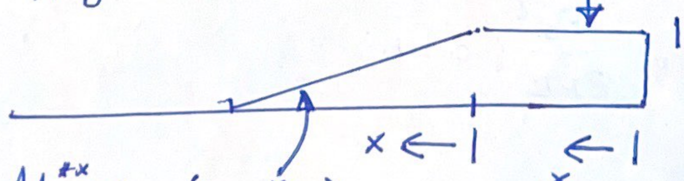
[1e] (5 pts) Use the method of **virtual forces** to show that rotation of the beam (measured clockwise) at D is:

$$\theta_D = \frac{Pa}{6EI}\left[2b + 3a\right] \tag{3}$$

Show all of your working.

Apply unit moment at D.

$M^{**}(x) = 1$



$M^{**}(x) = (1 - x/b)$

$$\theta_D = \int_0^a 1\cdot\frac{Px}{EI}\,dx + \int_0^b \frac{Pa}{EI}\left(1-\frac{x}{b}\right)^2 dx = \frac{Pa}{6EI}\left[2b+3a\right].$$

Question 2: 10 points

OPTIONAL: Derive Elastic Curve for Beam Deflection. Figure 2 is a front elevation view of a cantilevered beam carrying a single point load P. EI is constant along the beam structure A-B-C-D.

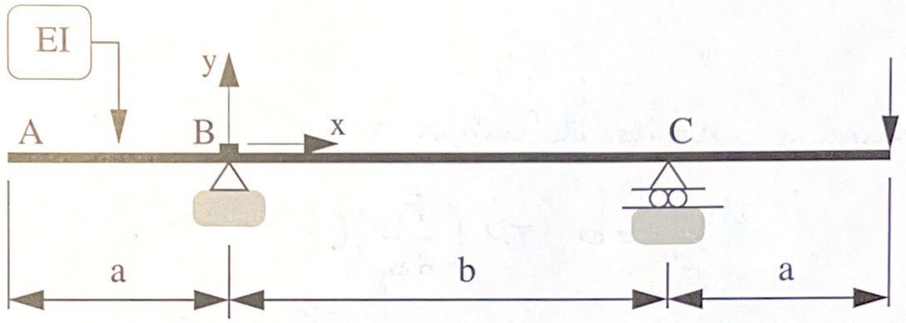


Figure 2: Cantilevered beam carrying a single applied load P.

[2a] (5 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{-M(x)}{EI}\right], \tag{4}$$

(notice the minus sign on M(x)) and appropriate boundary conditions, show that that vertical beam deflection along the beam segment B-C is:

$$y(x) = \frac{Pa}{6EIb}\left(b^2 - x^2\right)x. \tag{5}$$

From part [1b], $M(x) = -P\left(\frac{a}{b}\right)x$.

$\Rightarrow \frac{d^2y}{dx^2} = \frac{M(x)}{EI}$

$\Rightarrow EI\, dy/dx = -P\left(\frac{a}{b}\right)\frac{x^2}{2} + A$

$\Rightarrow EI\, y(x) = -P\left(\frac{a}{b}\right)\frac{x^3}{6} + Ax + B.$

B.C. $y(0) = 0, \Rightarrow B = 0$

$y(b) = 0, \Rightarrow A = \frac{Pab}{6}$

$\Big\}\; EI\,y(x) = \left(\frac{Pab}{6}x - \frac{Pa}{b}\cdot\frac{x^3}{6}\right)$

$\Rightarrow y(x) = \frac{Pa}{6EIb}\left(b^2 - x^2\right)x.$

[2b] (5 pts) Hence (i.e., using your results from part [2a] as a starting point), show that the maximum upwards deflection of the beam occurs at a distance $b/\sqrt{3}$ from B, and that its value is:

$$y_{\text{maximum upward}} = \frac{Pab^2}{9\sqrt{3}EI}. \tag{6}$$

Show all of your working.

Maximum upwords deflection occurs at

$$\frac{dy}{dx} = 0 \Rightarrow \left(\frac{Pa}{6b}\right)(b^2 - 3x^2) = 0$$

$$\Rightarrow x = \frac{b}{\sqrt{3}}. \quad \text{——— } (A)$$

Plug (A) into $y(x)$ gives:

$$y(b/\sqrt{3}) = \frac{Pa}{6EIb}\left(b^2 - \frac{b^2}{3}\right)\cdot\left(\frac{b}{\sqrt{3}}\right)$$

$$= \frac{Pab^2}{EI\, 9\sqrt{3}}.$$

Question 3: 10 points

OPTIONAL: Principle of Superposition and Method of Moment Area. Figure 3 is a front elevation view of a multi-span beam structure that carries uniform load w_o (N/m) along beam segment B-C. EI is constant along the beam structure A-B-C-D.

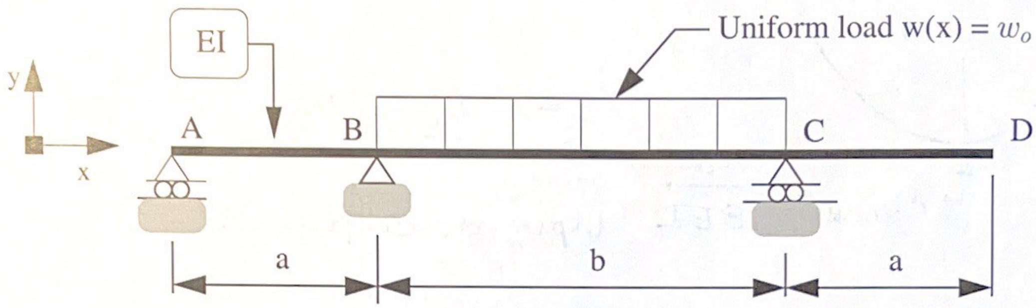


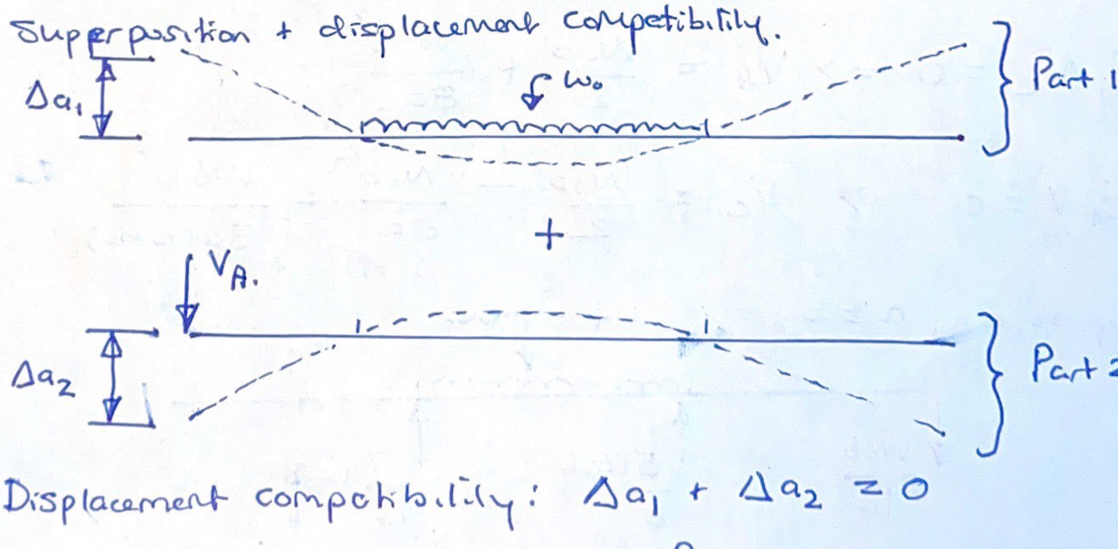
Figure 3: Multi-span beam carrying a uniform load on segment B-C.

[3a] (2 pts) Compute the degree of indeterminacy for this structure.

$\hat{i} = 1.$

[3b] (2 pts) Draw and label a diagram indicating how the **principle of superposition** can be used to simplify the analysis of the multi-span beam structure.

Here's one way – release the roller support at A. Apply superposition + displacement compatibility.

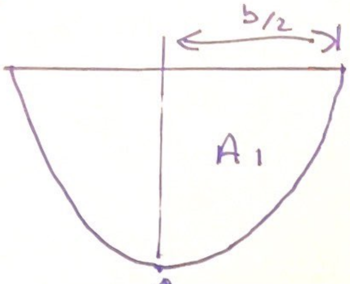


Displacement compatibility: $\Delta a_1 + \Delta a_2 = 0$

 $\Rightarrow$ find V_A.

[3c] (6 pts) Using the method of moment area, or otherwise compute the support reactions at supports A, B and C. Check that you reaction forces are in equilibrium with the applied loads.

M/EI



$$M_{max} = \frac{w_0 b^2}{8EI}$$

$$A_1 = \frac{2}{3} \cdot \frac{b}{2} \cdot \frac{w_0 b^2}{8EI} = \frac{w_0 b^3}{24EI}.$$

$$\Rightarrow \theta_B = \theta_C = \frac{w_0 b^3}{24EI}.$$

Upwards deflection: $\Delta a_1 = \theta_B \cdot a$

$$= \frac{w_0 a b^3}{24EI} \uparrow$$

From Question [1c]:

$$\Delta a_2 = \frac{V_A \cdot a^2(a+b)}{3EI} \downarrow$$

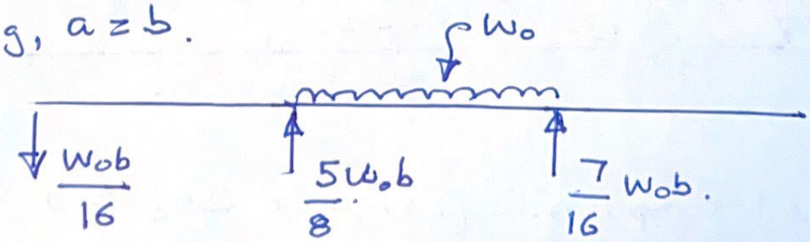
Compatibility of displacements; $\Delta a_1 \uparrow + \Delta a_2 \downarrow = 0.$

$$\Rightarrow V_A = \frac{w_0}{8} \cdot \frac{b^3}{a(a+b)} \downarrow$$

$$\sum M_C = 0 \Rightarrow V_B = \frac{w_0 b}{2} + \frac{w_0 b^2}{8a}.$$

$$\sum V = 0 \Rightarrow V_C = \frac{w_0 b}{2} - \frac{w_0 b^2}{8a} + \frac{w_0 b^3}{8a(a+b)}.$$

Suppose, e.g., $a = b$.



Note: $V_A + V_B + V_C = w_0 b$ ✓

Question 4: 10 points

OPTIONAL: Structural Analysis with Method of Virtual Displacements. The cantilevered beam structure shown in Figure 4 supports a uniform load distributed w(x) = w_o (N/m) over the beam section B-C-D.

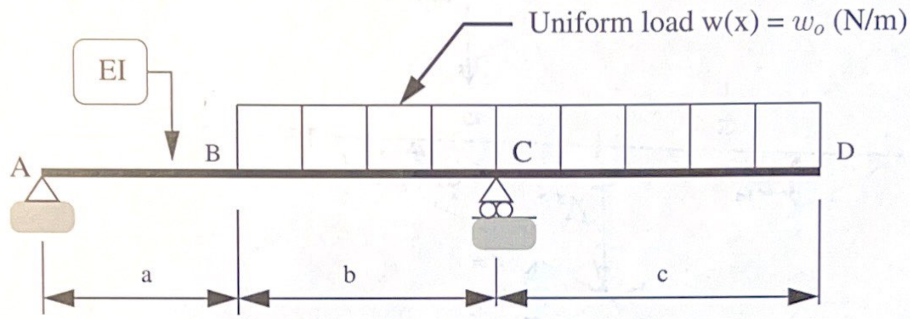
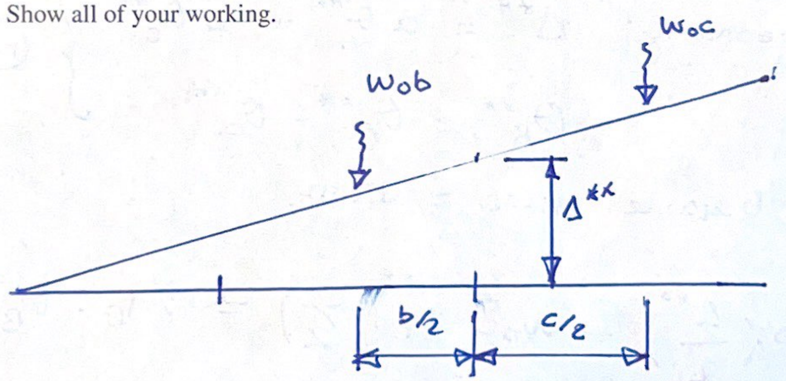


Figure 4: Front elevation view of a simple beam structure.

[4a] (4 pts) Use the method of **virtual displacements** to compute formulae for the vertical reactions at A and C. Show all of your working.



EWD = 0

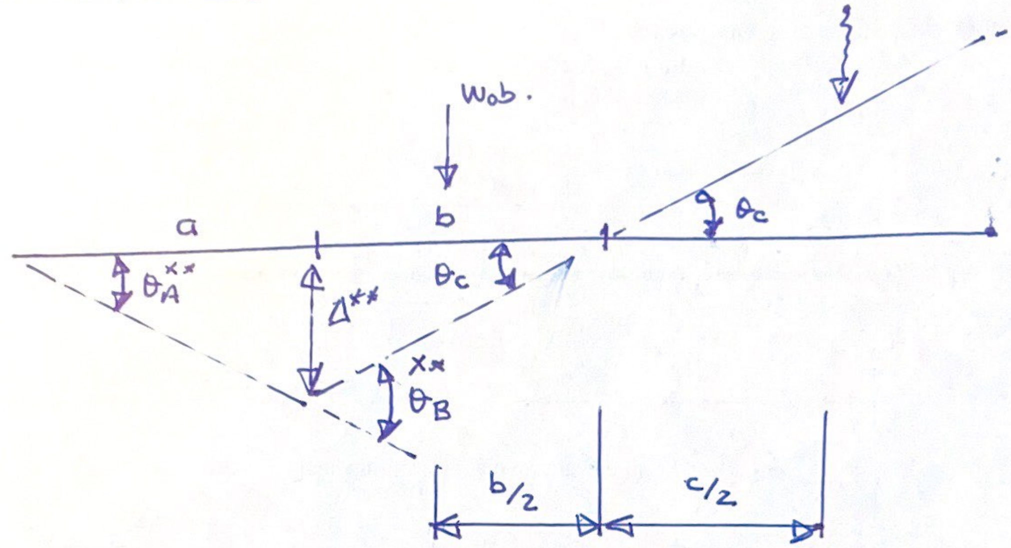
$$V_c \cdot \Delta^{**} + (-W_o b)\,\Delta^{**}\left[\frac{(a + b/2)}{a+b}\right] + (-W_o c)\,\Delta^{**}\left[\frac{a+b+c/2}{a+b}\right] = 0$$

$$\Rightarrow V_c = W_o b\left[\frac{a + b/2}{a+b}\right] + W_o c\left[\frac{a+b+c/2}{a+b}\right].$$

Similarly

$$V_A = W_o b\left[\frac{b/2}{(a+b)}\right] - W_o c\left[\frac{(c/2)}{(a+b)}\right].$$

[4b] (6 pts) Use the method of **virtual displacements** to compute a formula for the bending moment at B. Show all of your working.



From geometry: $\left.\begin{aligned} \Delta^{**} &= a\,\theta_A^{**} = b\,\theta_c^{**} \\ \theta_B^{**} &= \theta_A^{**} + \theta_c^{**} \end{aligned}\right\}$ (A)

Energy balance: EWD = IWD.

$$w_0 b\left(\frac{\Delta^{**}}{2}\right) - w_0 c\,\theta_c^{**}\left(\frac{c}{2}\right) = M_B \cdot \theta_B^{**} \quad \text{— (B)}$$

Plug equations (A) into (B), simplify:

$$M_B = \frac{w_0}{2}\,\frac{a}{(a+b)}\cdot\left[b^2 - c^2\right].$$

Note: If $b = c \Rightarrow M_B = 0$.

Question 5: 10 points

OPTIONAL: Use Principle of Virtual Work to Compute Displacements. Consider the articulated cantilever beam structure shown in Figure 5.

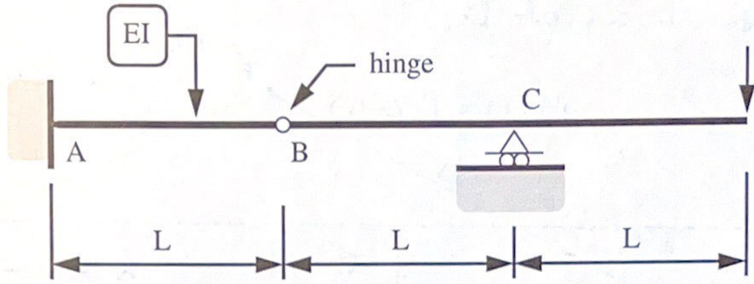
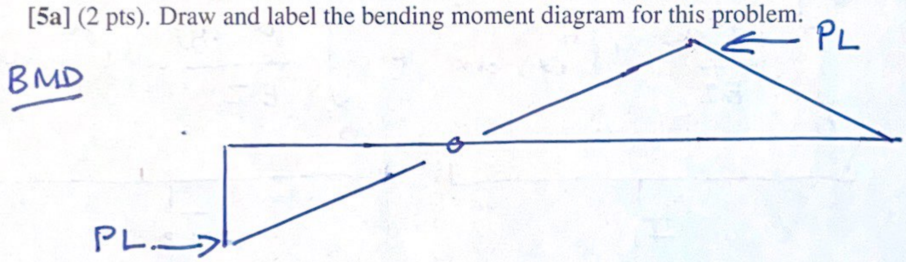


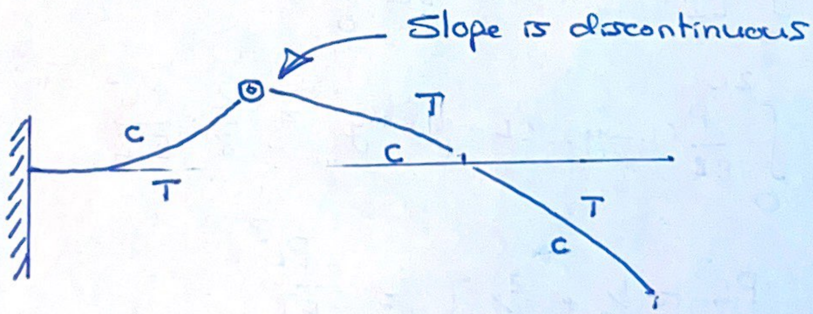
Figure 5: Elevation view of articulated cantilever beam structure.

At Point A, the cantilever is fully fixed (no movement) to a wall. Point B is a hinge. Both members have cross section properties EI. A single point load **P** (N) is applied at node D as shown in the figure.

[5a] (2 pts). Draw and label the bending moment diagram for this problem.



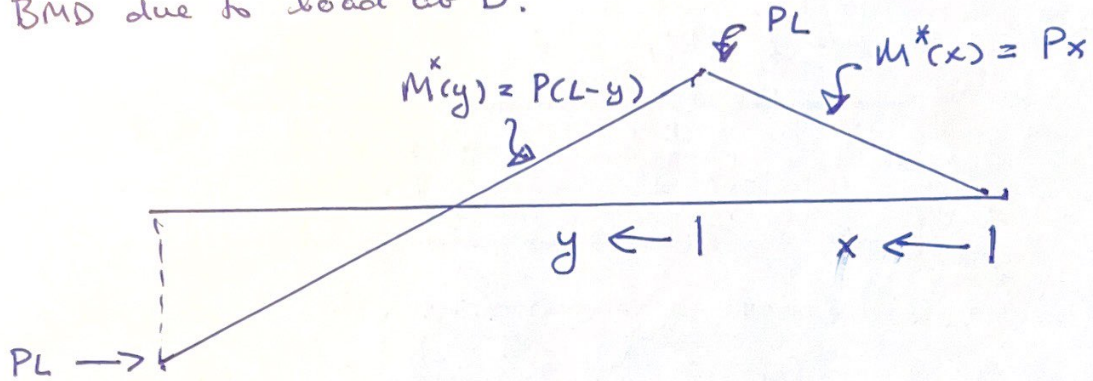
[5b] (2 pts). Qualitatively sketch the deflected shape. Indicate regions of tension/compression, and any points where slope of the beam is discontinuous.



[5c] (6 pts). Use the method of **virtual forces** to compute the **vertical displacement** and **end rotation** of the beam at D.

Show all of your working.

BMD due to load at D.



Deflection at D

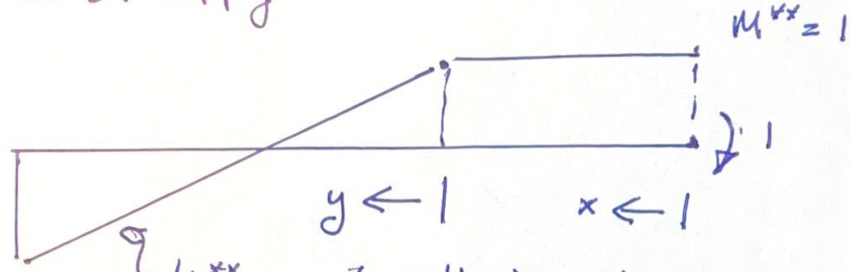
$$\Delta_D = \underbrace{\int_0^L \frac{M^*(x)}{EI} \cdot M^{**}(x)\, dx}_{I_1} + \underbrace{\int_0^{2L} \frac{M^*(y)}{EI} \cdot M^{**} dy}_{I_2}.$$

$$I_1 = \int_0^L \frac{Px}{EI} \cdot x\, dx = \frac{P}{EI} \int_0^L x^2 dx = \frac{PL^3}{3EI}.$$

$$I_2 = \int_0^{2L} \frac{P}{EI}(L-y).(L-y)\, dy = \frac{2}{3} \cdot \frac{PL^3}{EI}.$$

$$\Rightarrow \Delta_D = \frac{PL^3}{EI}\left[\frac{1}{3} + \frac{2}{3}\right] = \frac{PL^3}{EI}.$$

Rotation at D. Apply virtual moment at D.



$$M^{**}(y) = \left(1 - \frac{y}{L}\right) = \frac{1}{L}(L-y).$$

$$\theta_D = \underbrace{\int_0^L \frac{P.x.\,1}{EI}\,dx}_{I_1} + \underbrace{\int_0^{2L} \frac{P(L-y)}{EI}\cdot\frac{1}{L}\cdot(L-y)\,dy}_{I_2}.$$

$$I_1 = \frac{P}{EI}\int_0^L x\,dx = \frac{PL^2}{2EI}.$$

$$I_2 = \frac{P}{EI.L}\int_0^{2L}(L-y)^2\,dy = \frac{2}{3}\cdot\frac{PL^2}{EI}.$$

$$\Rightarrow \theta_D = \frac{PL^3}{EI}\left[\frac{1}{2} + \frac{2}{3}\right] = \frac{7}{6}\frac{PL^2}{EI},$$

Question 6: 10 points

OPTIONAL: Zero-Force Memebers, Virtual Work, and Flexibility Matrices. Consider the truss structure shown in Figure 6.

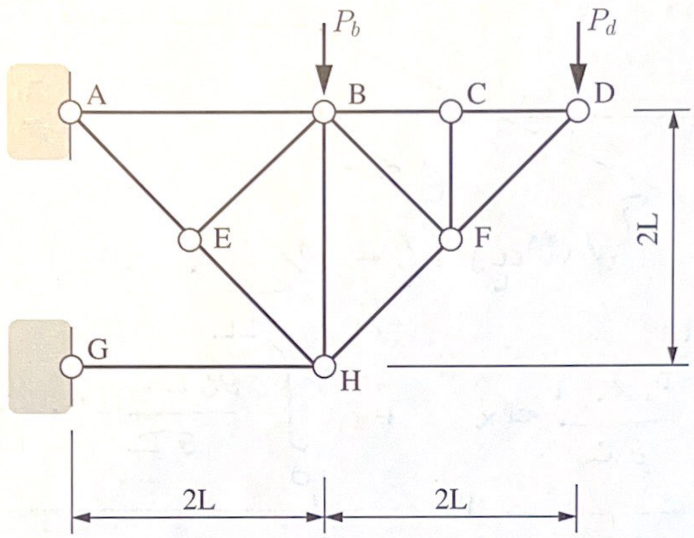
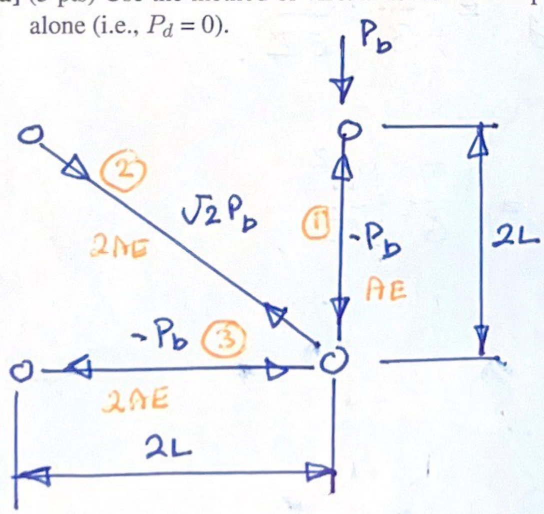


Figure 6: Elevation view of a pin-jointed truss.

The horizontal and vertical degrees of freedom are fully-fixed at supports A and G. The truss carries vertical loads P_b and P_d at nodes B and D, respectively. Frame members **AE**, **EH** and **GH** have members have cross section properties **2AE**. Otherwise, the members have cross section properties AE.

[6a] (3 pts) Use the method of **virtual forces** to compute the vertical deflection at node B due to load P_b alone (i.e., $P_d = 0$).

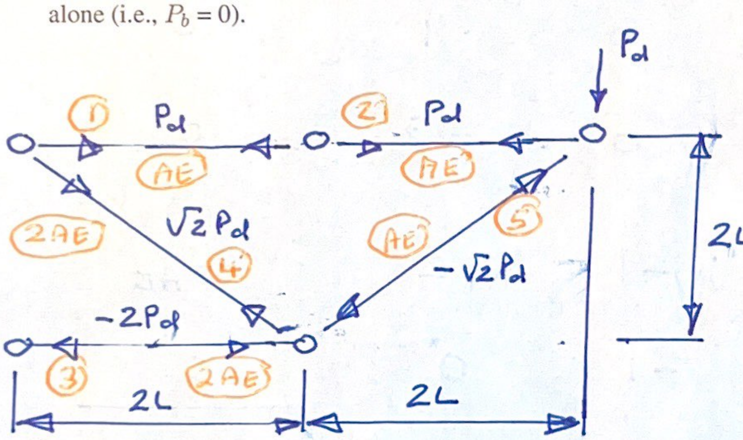


Question [6a] continued:

Member	L/AE	F_i	f_i	$\frac{L_i F_i f_i}{AE}$
1	$2L/AE$	$-P_b$	-1	$\frac{2P_b \cdot L}{AE}$
2	$\frac{\sqrt{2}L}{AE}$	$\sqrt{2}P_b$	$\sqrt{2}$	$\frac{2\sqrt{2}P_b \cdot L}{AE}$
3	$\frac{2L}{2AE}$	$-P_b$	-1	$\frac{P_b L}{AE}$

$$\frac{PL}{AE}\left[3 + 2\sqrt{2}\right].$$

[6b] (3 pts) Use the method of **virtual forces** to compute the vertical deflection at node D due to load P_d alone (i.e., $P_b = 0$).



$$\Delta_d = \sum_{i=1}^{5} \frac{F_i f_i L_i}{AE} = \left[\frac{P_d \cdot 1 \cdot 2L}{AE} + \frac{P_d \cdot 1 \cdot 2L}{AE} + \frac{-2P_d \cdot -1 \cdot 2L}{2AE} + \frac{\sqrt{2}P_d \cdot \sqrt{2} \cdot 2\sqrt{2}L}{2AE} + \frac{-\sqrt{2}P_d \cdot -\sqrt{2} \cdot 2\sqrt{2}L}{AE}\right]$$

$$= \frac{P_d \cdot L}{AE}\left[2 + 2 + 2 + 2\sqrt{2} + 4\sqrt{2}\right]$$

$$= \frac{P_d \cdot L}{AE}\left[6 + 6\sqrt{2}\right].$$

[6c] (4 pts) Use the method of **virtual forces** to compute the two-by-two flexibility matrix connecting the vertical displacements at points B and D to applied loads P_b and P_d, i.e., as a function of P_b, P_d, L and AE.

$$\begin{bmatrix} \triangle_b \\ \triangle_d \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_b \\ P_d \end{bmatrix}. \quad (7)$$

Member	L/AE	f_b	f_d	$\frac{f_b \cdot f_d \cdot L}{AE}$
AB	$2L/AE$	0	1	0
AH	$\frac{2\sqrt{2}L}{2AE}$	$\sqrt{2}$	$\sqrt{2}$	$\frac{2\sqrt{2}L}{AE}$
BD	$2L/AE$	0	1	0
DH	$\frac{2\sqrt{2}L}{AE}$	0	$-\sqrt{2}$	0
GH	$\frac{2L}{2AE}$	-1	-2	$2L/AE$
BH	$2L/AE$	-1	0	0
				$\frac{L}{AE}[2 + 2\sqrt{2}]$

Hence,

$$\begin{bmatrix} \Delta_b \\ \Delta_d \end{bmatrix} = \frac{L}{AE} \begin{bmatrix} (3 + 2\sqrt{2}) & (2 + 2\sqrt{2}) \\ (2 + 2\sqrt{2}) & (6 + 6\sqrt{2}) \end{bmatrix} \begin{bmatrix} P_b \\ P_d \end{bmatrix}.$$