Department of Civil and Environmental Engineering, Spring Semester, 2021

ENCE 353 Final Exam, Open Notes and Open Book

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Exam Format and Grading. This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

Answer Question 1. Then answer **three of the five** remaining questions. Partial credit will be given for partially correct answers, so please show all your working.

IMPORTANT: Only the first four questions that you answer will be graded, so please **cross out the two questions you do not want graded** in the table below.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	50	

Question 1: 20 points

COMPULSORY: Moment-Area, Virtual Forces, Superposition. Figure 1 is a front elevation view of a simple beam structure carrying two external loads P. The beam has section properties EI near the supports and 2EI in the center section.

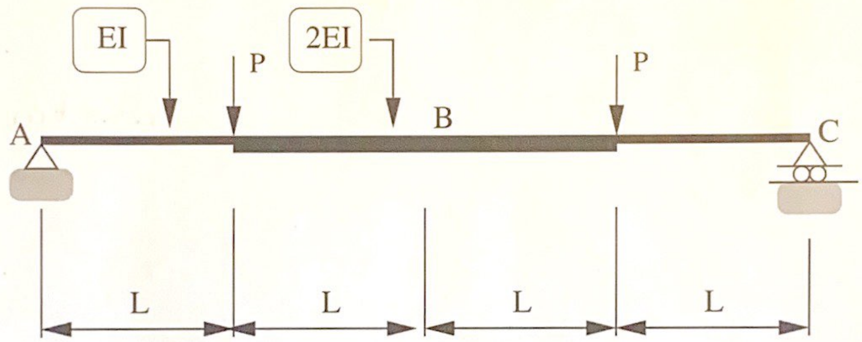
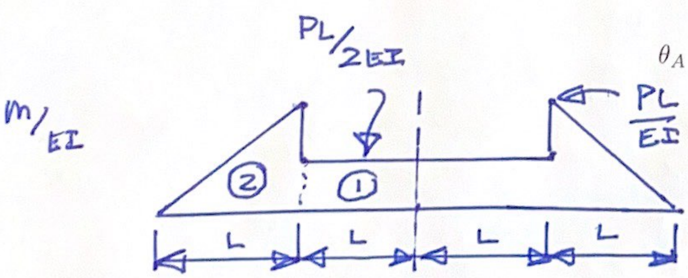


Figure 1: Simple beam structure (symmetric loads P).

[1a] (4 pts) Use the method of **MOMENT AREA** to show that the end rotation at A (measured clockwise) is:

$$\theta_A = \frac{PL^2}{EI}. \tag{1}$$



$$\theta_A - \theta_B = \int_0^{2L} \frac{M}{EI}\,dx.$$

$$= \frac{1}{2}\frac{PL^2}{EI} + \frac{PL}{2EI}\cdot L$$

$$= \frac{PL^2}{EI}$$

[1b] (4 pts) Use the method of **MOMENT AREA** to show that the vertical beam deflection at B is:

$$\triangle_B = \frac{13}{12}\frac{PL^3}{EI}. \tag{2}$$

$$A_1 = \frac{PL^2}{2EI}, \quad \bar{x}_1 = \frac{3}{2}L$$

$$A_2 = \frac{\frac{1}{2}PL^2}{EI}, \quad \bar{x}_2 = \frac{2}{3}L$$

$$\Delta_B = A_1\bar{x}_1 + A_2\bar{x}_2$$

$$= \frac{PL^2}{2EI}\cdot\frac{3}{2}L + \frac{PL^2}{2EI}\cdot\frac{2}{3}L$$

$$= \frac{13}{12}\cdot\frac{PL^3}{EI}.$$

[1c] (4 pts) A function is said to be even if it has the property f(x) = f(-x) (i.e., it is symmetric about the y axis). And a function is said to be odd if it has the property g(x) = - g(-x) (i.e., it is skew-symmetric about the y axis). One example of an even function is cos(x), and one example of an odd function is sin(x).

Using high-school-level calculus (or otherwise), show that:

$$\int_{-h}^{h} f(x)g(x)dx = 0. \tag{3}$$

Please show all of your working.

Split integral into two parts:

$$\int_{-h}^{h} f(x)g(x)\,dx = \int_{-h}^{0} f(x)g(x)\,dx + \underbrace{\int_{0}^{h} f(x)g(x)\,dx}_{I}.$$

Let $y = -x$, $dy = -dx$ $\Big\} \rightarrow$ $f(x) = f(y)$, $g(x) = -g(y)$.

$$\Rightarrow \int_{-h}^{0} f(x)g(x)\,dx = -\int_{-h}^{0} f(y)g(y)\,dy = -I$$

$$\Rightarrow \int_{-h}^{h} f(x)g(x)\,dx = -I + I = 0.$$

Figure 2 shows the same beam structure, but now the external loads are rearranged so that one load points down and one load points up.

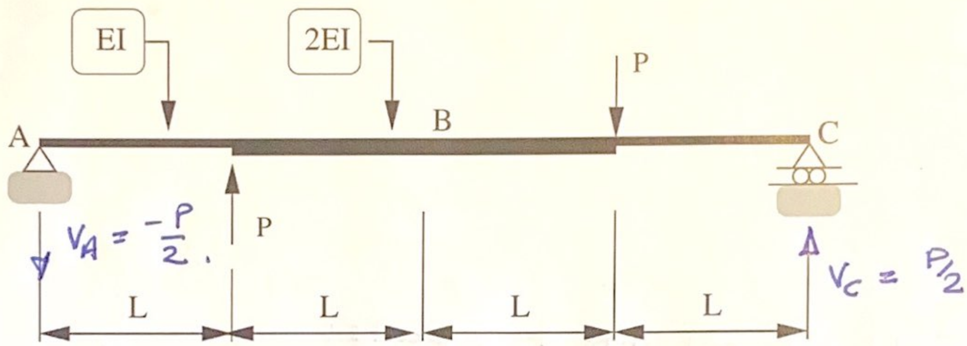
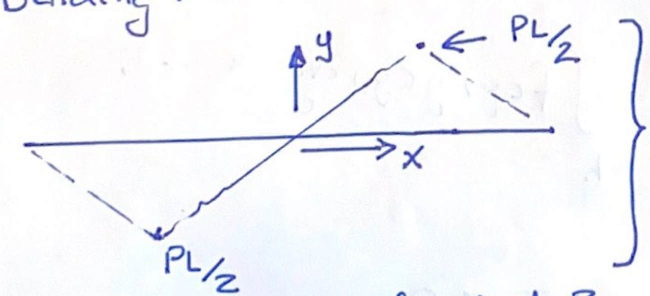


Figure 2: Simple beam structure (skew-symmetric loads P).

[1d] (4 pts) Use the method of **VIRTUAL FORCES** and a coordinate system positioned at B to show that the vertical displacement of B is zero, i.e., $\triangle_B = 0$.

$$\left.\begin{array}{l} \sum_1 M_A = 0 \Rightarrow PL - 3PL + V_c(4L) = 0 \\ \sum_1 F_y = 0 \Rightarrow P - P + V_A + V_c = 0 \end{array}\right\} \begin{array}{l} V_C = P/2 \\ V_A = -P/2 \end{array}$$

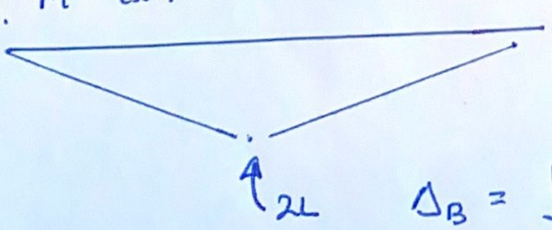
Bending Moment due to real loads $M^*(x)$



BMD is skew-symmetric

Apply virtual unit load at B.

BMD. $M^{**}(x)$.



BMD is symmetric.

$$\Delta_B = \int_{-2L}^{2L} \frac{M^*}{EI} \cdot M^{**} dx = 0.$$

Now consider the problem.

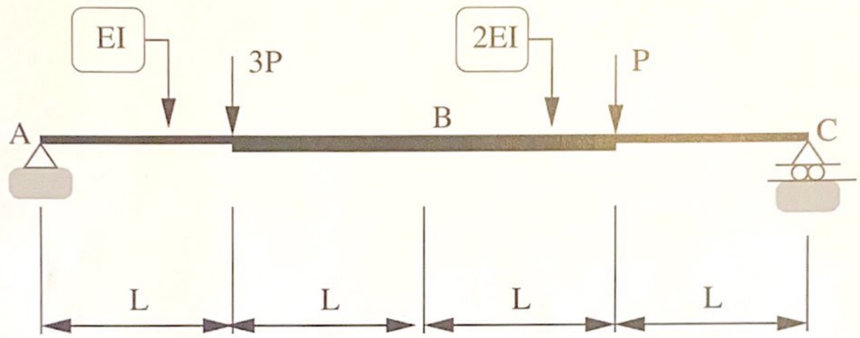
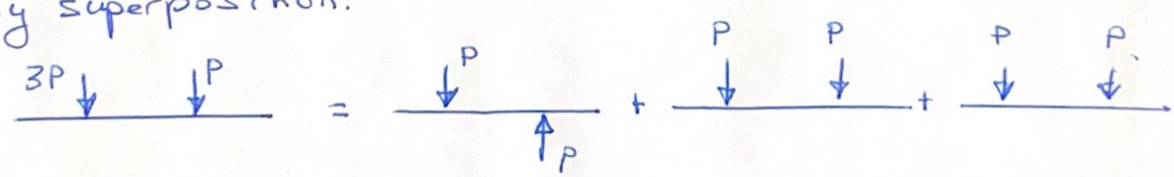


Figure 3: Simple beam structure with external loads 3P and P.

[1e] (4 pts) Use your answers from parts [1b] and [1d] to write down an expression for the vertical deflection at B due to the loading pattern shown in Figure 3.

Note: You should find this is a one line answer.

By superposition:



$$\Delta_B = 0 + 2\left(\frac{13}{12}\frac{PL^3}{EI}\right) = \frac{13}{6}\frac{PL^3}{EI}.$$

Question 2: 10 points

OPTIONAL: Computing Displacements with the Method of Virtual Forces. Figure 4 is a front elevation view of a dog-leg cantilever beam carrying a clockwise moment M (N.m) at point C.

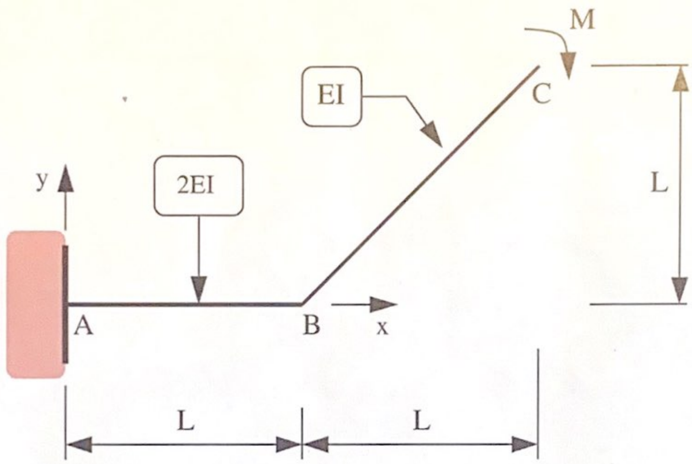


Figure 4: Dog-leg cantilever beam carrying end moment M (N.m).

The flexural stiffness 2EI is constant along A-B, and EI along B-C. The axial stiffness EA is very high and, as such, axial displacements can be ignored in the analysis.

[2a] (5 pts) Use the method of **VIRTUAL FORCES** to show that the clockwise rotation of the beam at point C is:

$$\theta_c = \frac{ML}{EI}\left[\frac{1+2\sqrt{2}}{2}\right]. \tag{4}$$

$$\theta_c = \int_A^B \frac{M}{2EI}\,dx + \int_B^C \left(\frac{M}{EI}\right) dx$$

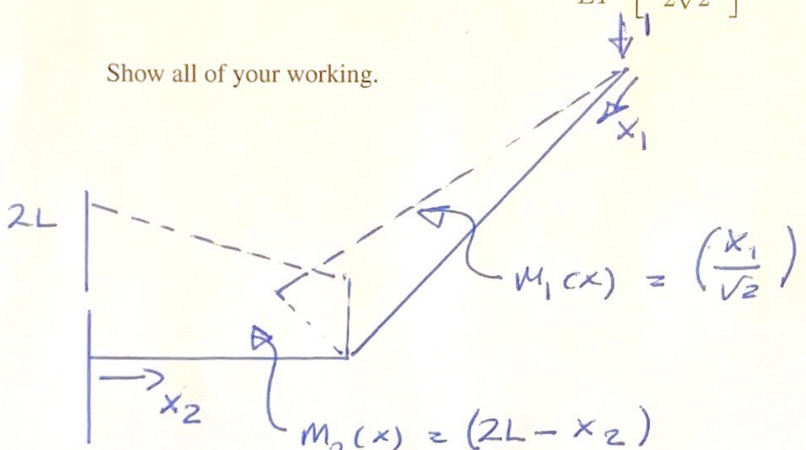
$$= \frac{ML}{2EI} + \frac{\sqrt{2}L\,M}{EI}$$

$$= \left(\frac{ML}{EI}\right)\left[\frac{1+2\sqrt{2}}{2}\right].$$

[2b] (5 pts) Use the method of **VIRTUAL FORCES** to show that the vertical displacement at C (measured downwards) is:

$$y_c = \frac{ML^2}{EI}\left[\frac{2+\sqrt{2}}{2\sqrt{2}}\right]. \tag{5}$$

Show all of your working.



$$y_c = \int_0^{\sqrt{2}L} \left(\frac{M}{EI}\right)\left(\frac{x_1}{\sqrt{2}}\right) dx_1 + \int_0^{L} \frac{M}{2EI}(2L - x_2)\, dx_2$$

$$= \frac{M}{EI}\left[\frac{1}{\sqrt{2}}\,\frac{1}{2}\,x_1^2\right]_0^{\sqrt{2}L} + \frac{M}{2EI}\left[2Lx_2 - \frac{1}{2}x_2^2\right]_0^L$$

$$= \frac{ML^2}{EI}\left[\frac{1}{\sqrt{2}} + \frac{1}{2}\right]$$

$$= \frac{ML^2}{EI}\left[\frac{2+\sqrt{2}}{2\sqrt{2}}\right].$$

Question 3: 10 points

OPTIONAL: Structural Analysis with Method of Virtual Displacements. The cantilevered beam structure shown in Figure 5 supports a point load P (N) at B and a uniformly distributed load w (N/m) between points C and D.

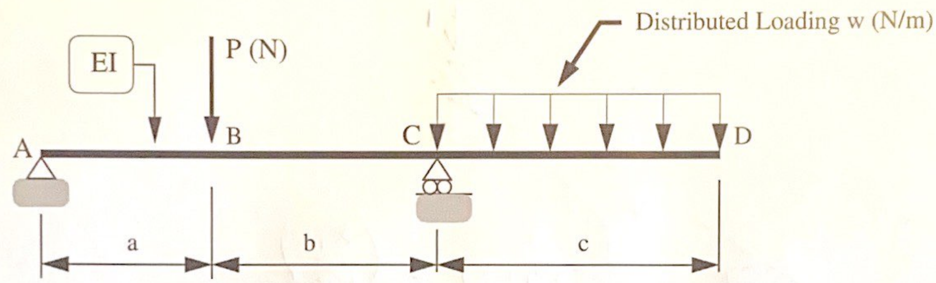
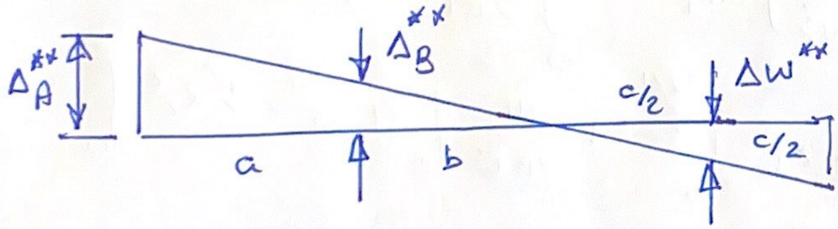


Figure 5: Front elevation view of a simple beam structure.

[3a] (4 pts) Use the method of **VIRTUAL DISPLACEMENTS** to compute formulae for the vertical reactions at A and C. Show all of your working.

Virtual Displacement at A,



From geometry:

$$\Delta_B^{**} = \frac{b}{(a+b)} \Delta_A^{**}$$

$$\Delta_w^{**} = \frac{c}{2(a+b)} \Delta_A^{**}$$

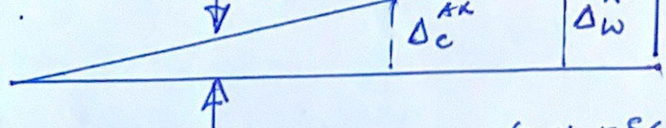
$$IWD = EWD = 0$$

$$EWD = V_A \Delta_A^{**} - \frac{Pb\,\Delta_A^{**}}{(a+b)} + \frac{wc^2 \Delta_A^{**}}{2(a+b)} = 0$$

$$\Rightarrow V_A = \frac{Pb}{a+b} - \frac{wc^2}{2(a+b)}$$

$$0 = \Delta_C^{**} V_C - P\Delta_B^{**} - wc\,\Delta_C^{**}$$

Virtual Displacement at C



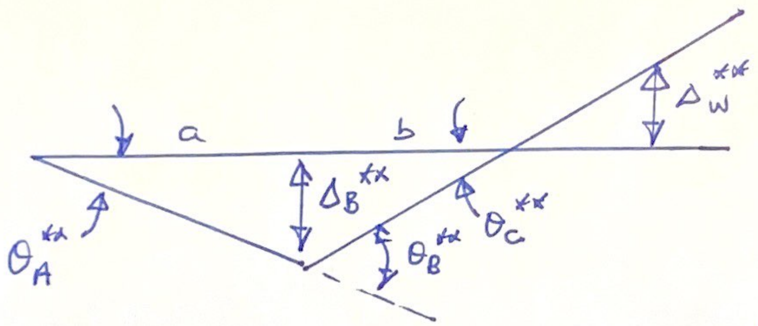
$$\Rightarrow V_C = \frac{Pa}{(a+b)} + \frac{wc(a+b+c/2)}{(a+b)}$$

$$\Delta_B^{**} = \frac{a}{a+b} \Delta_C^{**}; \quad \Delta_W^{**} = \frac{(a+b+c/2)}{(a+b)} \Delta_C^{**}$$

Note: $V_A + V_C = P + wc$ ✓

[3b] (6 pts) Use the method of **VIRTUAL DISPLACEMENTS** to compute a formula for the bending moment at B. Show all of your working.

Apply virtual rotation at B.



From geometry: $\theta_B^{**} = \theta_A^{**} + \theta_C^{**} = \dfrac{\Delta_B^{**}}{a} + \dfrac{\Delta_B^{**}}{b}.$

$$\left.\begin{aligned} \Delta_B^{**} &= a\,\theta_A^{**} \\ \Delta_B &= b\,\theta_C^{**} \end{aligned}\right\}$$

$$\Delta_W^{**} = \frac{c}{2b}\Delta_B^{**}$$

IWD = EWD

$$\Rightarrow M_B\,\theta_B^{**} = P\,\Delta_B^{**} - Wc.\,\Delta_W^{**}$$

$$\Rightarrow M_B\left(\frac{\Delta_B^{**}}{a} + \frac{\Delta_B^{**}}{b}\right) = P\Delta_B^{**} - \frac{Wc^2}{2b}\Delta_B^{**}$$

$$\Rightarrow M_B = \frac{ab}{(a+b)}\left[P - \frac{Wc^2}{2b}\right] \checkmark$$

Note: $M_B = V_A \cdot a$ ✓

Question 4: 10 points

OPTIONAL: Principle of Virtual Work. Figure 6 is a front elevation view of a simple truss that supports vertical loads at nodes C and D. All of the truss members have cross section properties AE except B-C which has section properties kAE; here $k > 0$.

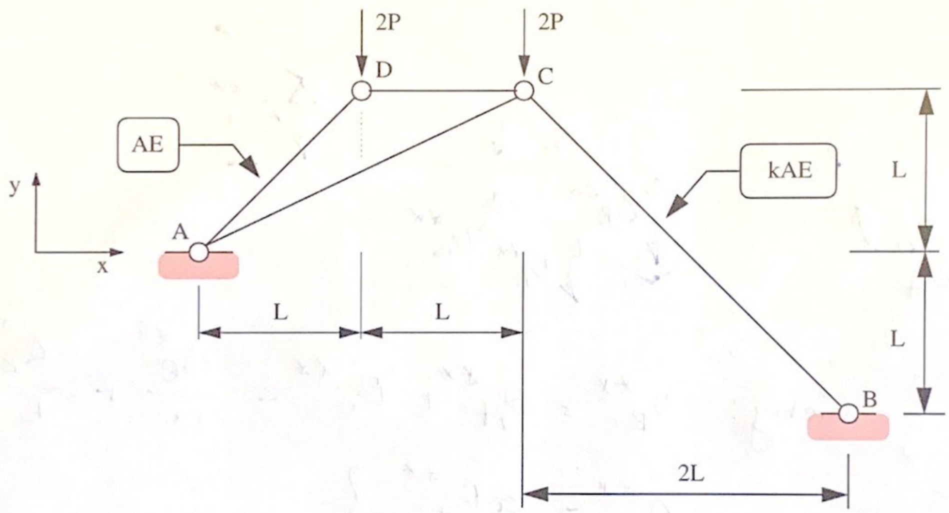
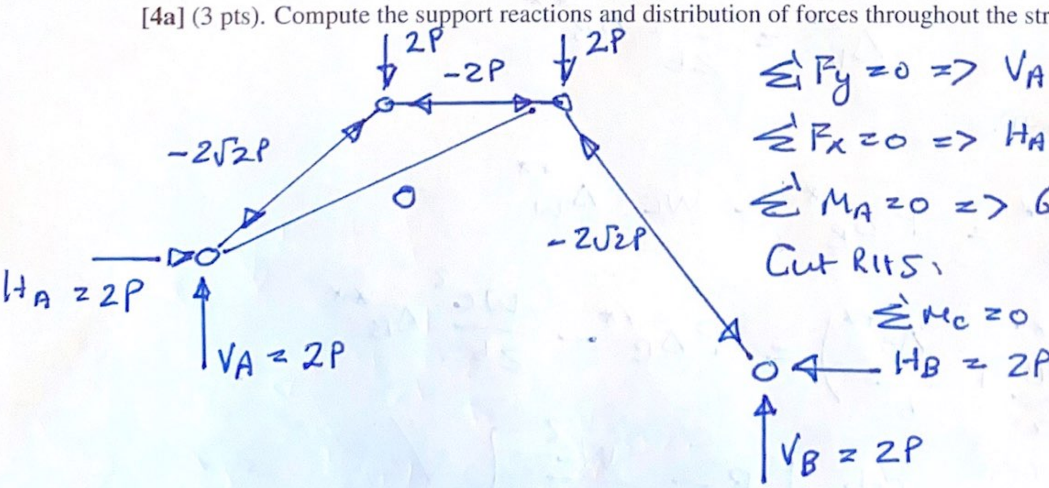


Figure 6: Front elevation view of a simple truss.

[4a] (3 pts). Compute the support reactions and distribution of forces throughout the structure.



$\sum F_y = 0 \Rightarrow V_A + V_B - 4P = 0$

$\sum F_x = 0 \Rightarrow H_A - H_B = 0$

$\sum M_A = 0 \Rightarrow 6P = 4V_B - H_B$

Cut RHS,

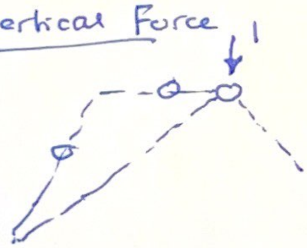
$\sum M_C = 0 \Rightarrow V_B = H_B.$

[4b] (7 pts). Use the method of **VIRTUAL FORCES** to show that the total displacement at node C is:

$$\Delta = \frac{PL}{kAE}\left[\frac{8\sqrt{10}}{3}\right]. \tag{6}$$

Apply unit loads at C

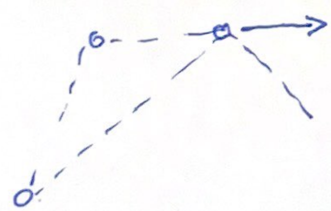
Vertical Force



$$\overline{AC} = \frac{\sqrt{5}}{2\sqrt{2}}\,\overline{BC}$$

$$\overline{BC} = \frac{-2\sqrt{2}}{3}P$$

Horizontal force



$$\sum F_y = 0 \Rightarrow \overline{AC} = \frac{-\sqrt{5}}{\sqrt{2}}\,\overline{BC}$$

$$\overline{BC} = -\frac{\sqrt{2}}{3}.$$

Displacements: $\Delta^* = \dfrac{\sum F f L}{EA}.$

$$\Delta_v = \frac{(-2\sqrt{2}P)(-2\sqrt{2}/3)\,2\sqrt{2}L}{kAE} = \frac{16\sqrt{2}}{3}\frac{PL}{kAE}.$$

$$\Delta_h = \frac{(2\sqrt{2}P)(\sqrt{2}/3)(2\sqrt{2}L)}{kAE} = \frac{8\sqrt{2}}{3}\frac{PL}{kAE}.$$

$$\Delta_{total} = (\Delta_v^2 + \Delta_h^2)^{1/2} = \frac{8\sqrt{10}}{3}\frac{PL}{kAE}.$$

Question 5: 10 points

OPTIONAL: Virtual Work and Flexibility Matrices. Figure 7 is a front elevation view of a simple truss that supports vertical loads P_c and P_d at nodes C and D. All of the truss members have cross section properties AE.

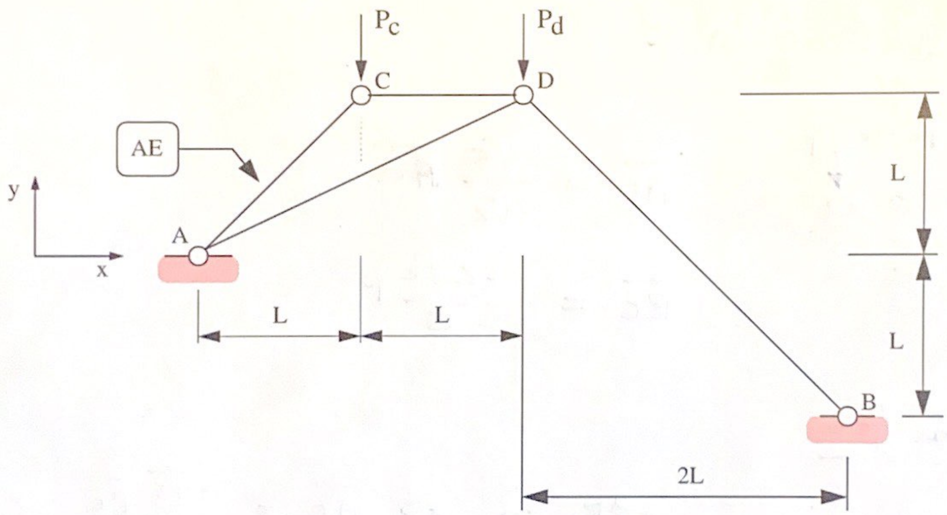
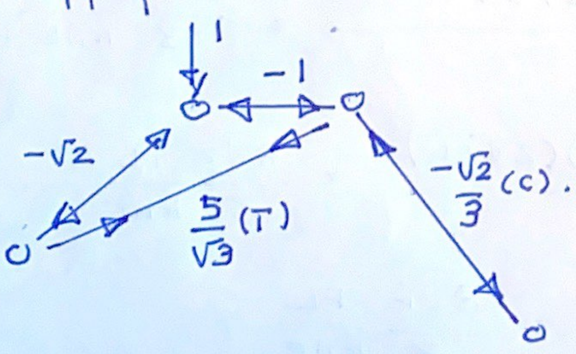


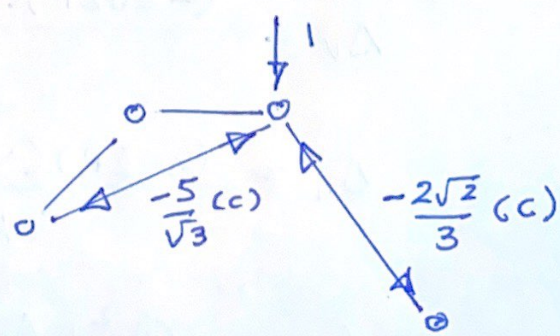
Figure 7: Front elevation view of a simple truss.

Use the principle of **VIRTUAL FORCES** to compute the two-by-two flexibility matrix connecting vertical displacements at points C and D to applied loads P_c and P_d, i.e.,

$$\begin{bmatrix} \Delta_c \\ \Delta_d \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_c \\ P_d \end{bmatrix}. \tag{7}$$

Apply unit forces at C & D.





Question 5 continued ...

Flexibility coefficients:

$$f_{11} = \sum_1 \frac{f_1^2 L}{AE} \; ; \; f_{22} = \sum_1 \frac{f_2^2 L}{AE} \; ; \; f_{12} = \sum_1 \frac{f_1 f_2 L}{AE}.$$

Tabular format:

Member	L	f_1	f_2	$f_1^2 L$	$f_2^2 L$	$f_1 f_2 L$
CA	$\sqrt{2}L$	$-\sqrt{2}$	0	$2\sqrt{2}$	0	0
CD	L	-1	0	1	0	0
AD	$\sqrt{5}L$	$\frac{\sqrt{5}}{3}$	$-5/\sqrt{3}$	$\frac{5}{9}\sqrt{5}$	$+\frac{25\sqrt{5}}{3}$	$-25/3\sqrt{3}$
BD	$2\sqrt{2}L$	$-2/\sqrt{3}$	$-\frac{2\sqrt{2}}{3}$	$\frac{8\sqrt{2}}{3}$	$16\sqrt{2}/3$	$16/3\sqrt{3}$.
				f_{11}	f_{22}	f_{12}.

$$f_{11} = 2\sqrt{2} + \frac{5}{9}\sqrt{5} + \frac{8\sqrt{2}}{3} = \frac{18\sqrt{2} + 5\sqrt{5} + 24\sqrt{2}}{9} = 7.84$$

$$f_{22} = \frac{25\sqrt{5}}{3} + \frac{16\sqrt{2}}{3} = \frac{25\sqrt{5} + 16\sqrt{2}}{3} = 26.17$$

$$f_{12} = -\sqrt{3}. = -1.73.$$

<u>Result:</u>

$$f = \begin{bmatrix} 7.84 & -1.73 \\ -1.73 & 26.17 \end{bmatrix}$$

Question 6: 10 points

OPTIONAL: Member forces in a Propped Cantilever. Figure 8 is an elevation view of a propped cantilever structure that carries external point loads P at points B and C. The cantilever support is fully fixed at point A.

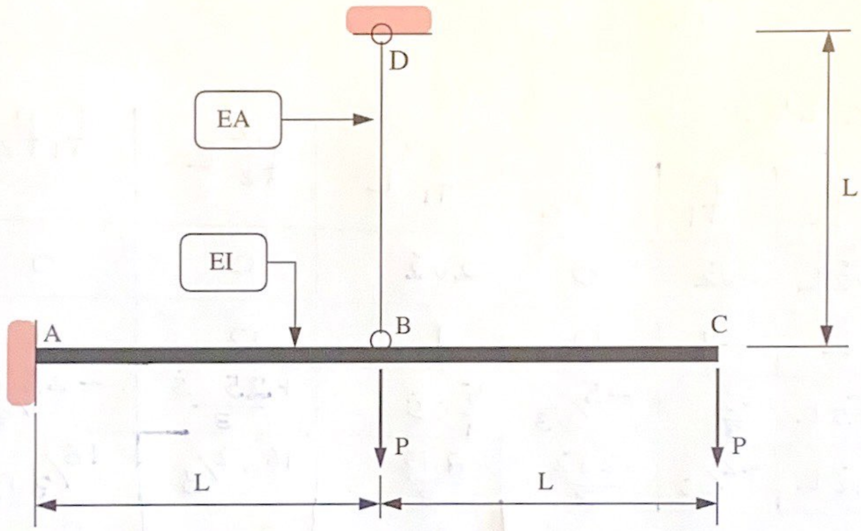


Figure 8: Elevation view of a propped cantilevel beam.

The structural system has constant section properties EI along the beam, and is supported by a truss element having section properties EA.

[6a] (1 pt) Compute the degree of indeterminacy for the propped cantilever beam.

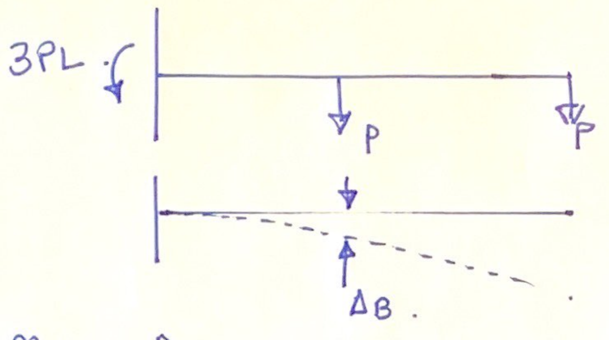
$\hat{i} = 3n - r = 1.$

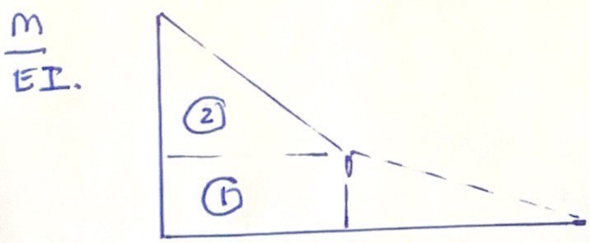
[6b] (5 pt) Using the method of moment-area, or otherwise, show that the axial force T in the truss element is related to the externally applied loads P by the equation:

$$\frac{7}{6}\frac{PL^3}{EI} = \frac{TL^3}{3EI} + \frac{TL}{EA}. \tag{8}$$

<u>Strategy:</u> Release T, then impose compatibility of displacements.

Question 6b continued ...





$$A_1 = \frac{PL^2}{EI}, \quad \bar{x}_1 = L/2$$

$$A_2 = \frac{PL^2}{EI}, \quad \bar{x}_2 = \frac{2}{3}L$$

$$\Delta_B = A_1 \bar{x}_1 + A_2 \bar{x}_2$$

$$= \frac{PL^2}{EI} \cdot \frac{L}{2} + \frac{PL^2}{EI} \cdot \frac{2}{3}L$$

$$= \frac{7}{6} \frac{PL^3}{EI}.$$

Displacement of B ~~due~~ due to tension T in cable:

$$\Delta_T = \frac{1}{3} \frac{TL^3}{EI}.$$

Extension of cable: $\Delta_c = \frac{TL}{AE}$.

Compatibility of displacements:

$$\Delta_B = \Delta_T + \Delta_c$$

$$\Rightarrow \frac{7}{6} \frac{PL^3}{EI} = \frac{TL^3}{3EI} + \frac{TL}{AE}.$$

[6c] (4 pt) Explain how the value of bending moment at the cantilever support (i.e., at point A) will change as a function of the problem parameters (i.e., P, T, E, I and A).

$M_A = 3PL - TL$.

Trends: As $P \uparrow \rightarrow M \uparrow$

As $T \uparrow \rightarrow M \downarrow$

As $E \uparrow$, M stays the same (impacts P & T equally)