Statically Determinate Structures

Mark A. Austin

University of Maryland

austin@umd.edu

ENCE 353, Fall Semester 2020

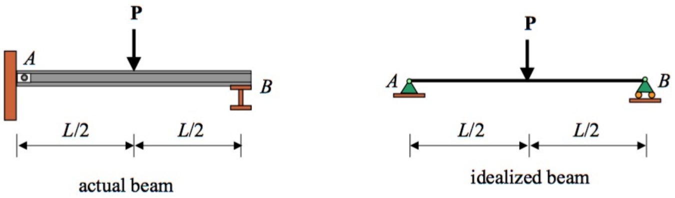
September 17, 2021

Overview

Introduction

Quick Review

Real-World and Idealized Abstractions



Statically Determinate Structure

- Can use statics to determine reactions and distribution of element-level forces.

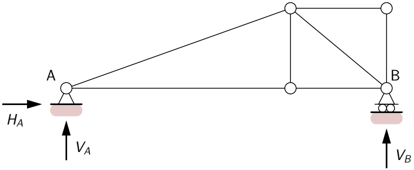
Statically Indeterminate Structure

- Statics alone are not enough to find reactions. Need to find additional information (e.g., material behavior).

Need for Mathematical Test

Three cases to consider:

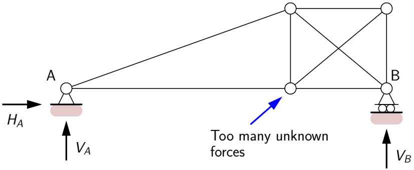
Test Structure A: Determinate.



Can compute:

- Support reactions. ✓
- Member forces. ✓

Test Structure B: Indeterminate.



Can compute:

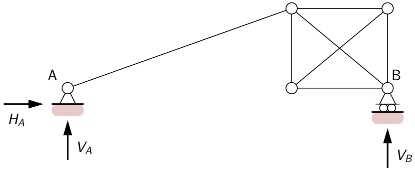
- Support reactions. ✓
- Member forces. ✗

Need for Mathematical Test

Test Structure C: Unstable.

Can compute:

- Support reactions. ✗
- Member forces. ✗

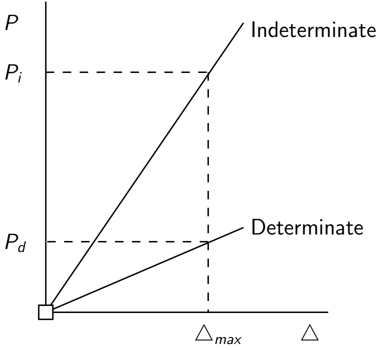


Key Points:

- Intuition on notions of determinacy will not scale. We need a mathematical test to classify structures.
- Initial inclination is to design for A and avoid B – it's complicated and probably won't work. Unless, there are benefits to B?

Benefits of Indeterminacy

Generally, indeterminate structures are stiffer than determinate structures.

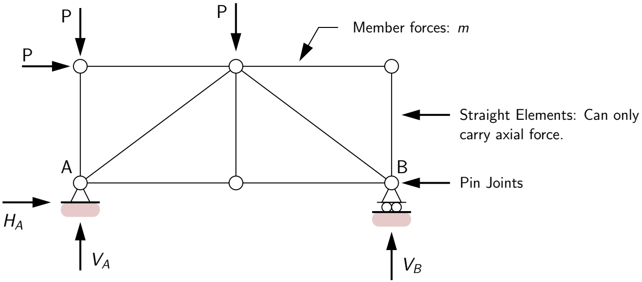


Materials such as steel/concrete are displacement constrained.

For a maximum allowable displacement ($\triangle_{max}$), the load carrying capacity of indeterminate structures (P_i) is greater than determinate structures (P_d).

Statical Determinacy of Trusses

Trusses



Formulae: If the truss has j joints $\rightarrow$ 2j equations of equilibrium.

$$\sum F_x = 0, \quad \sum F_y = 0. \tag{1}$$

Unknowns: No of reactions r, and no of member forces m.

Determinacy of Trusses

Test covers three categories:

- Truss is statically determinate: $m + r = 2j$.
- If $m + r < 2j \leftarrow$ Truss is unstable.
- If $m + r > 2j \leftarrow$ Truss is statically indeterminate.

Note. Tests are necessary but not sufficient.

For our three test cases:

Test Structure A: $r = 3$, $m = 7$, and $j = 5$.

- $m + r - 2j = 0 \rightarrow$ statically determinate.

Test Structure B: $r = 3$, $m = 8$, and $j = 5$.

- $m + r - 2j > 0 \rightarrow$ statically indeterminate.

Determinacy of Trusses

Test Structure C: $r = 3$, $m = 7$, and $j = 5$.

- $m + r$ - $2j = 0 \rightarrow$ statically determinate?

Bottom Line:

- Last test says statically determinate, but actually the test is faulty because structure is unstable.

Statical Determinacy of Planar Structures

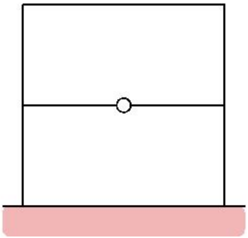
Planar Frame Structures

Three equations of equilibrium for each free body diagram:

If structure has n members and r unknown reactions,

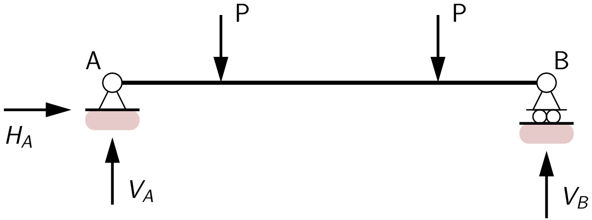
Test:

- If $r = 3n \rightarrow$ statically determinate.
- If $r > 3n \rightarrow$ statically indeterminate.
- If $r < 3n \rightarrow$ structure is unstable.



Planar Structures

Example 1.

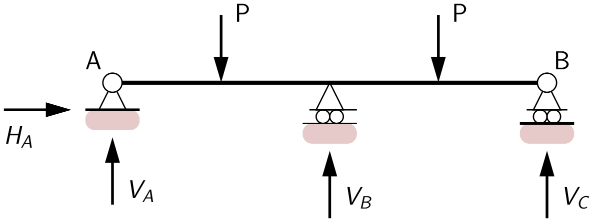


$n = 1$. r $= \{H_A, V_A, V_B\} = 3$.

Test: r - 3n $= 0 \rightarrow$ statically determinate.

Planar Structures

Example 2.

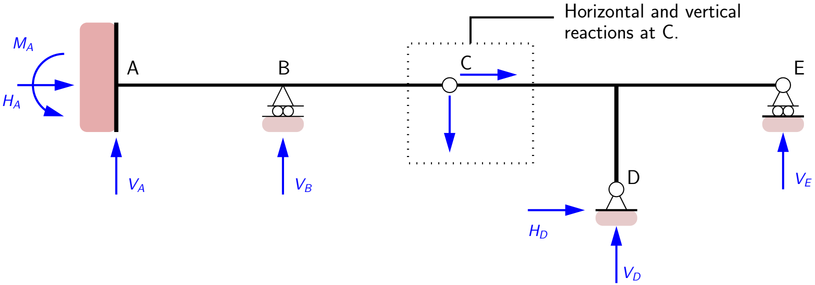


$n = 1$. r $= \{H_A, V_A, V_B, V_C\} = 4$.

Test: r - 3n $= 1 > 0 \rightarrow$ statically indeterminate.

Planar Frame Structures

Example 3.



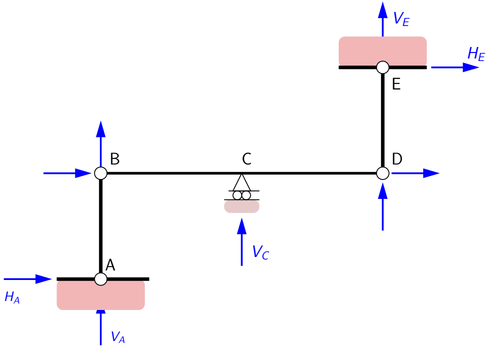
Two members: $n = 2$.

No reactions r $= \{H_A, V_A, M_A, \cdots, V_E\} = 9$.

Test: r - 3n $= 3 > 0 \rightarrow$ statically indeterminate to degree 3.

Planar Frame Structures

Counter Example 4. Example demonstrates test is necessary but not sufficient.

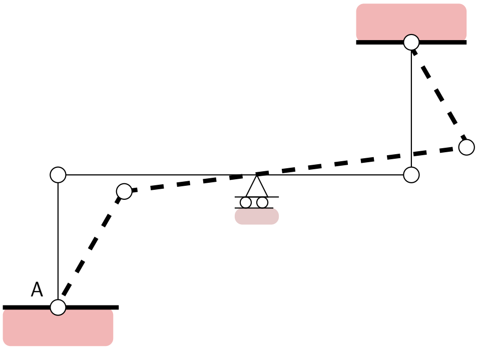


Three members: $n = 3$. No reactions r $= \{H_A, V_A, \cdots, H_E\} = 9$.

Test: r - 3n $= 0 \rightarrow$ statically determinate.

Planar Frame Structures

But this configuration is also a mechanism, i.e.,



Conclusion: Test is necessary but not sufficient!

Indeterminacy of Beams

Computing Degree of Indeterminacy

Definition. The degree of indeterminacy is equal to the number of additional equations needed to solve a problem uniquely.

Additional info:

- Compatibility of deformations – this is the force method.
- Equilibrium of forces – this is the displacement method.

Beams: $\hat{i}$ = f -3 - r, where:

- f = total no of external forces,
- r = total no of releases (hinges),
- 3 = no of equations from statics.

Indeterminacy of Beams

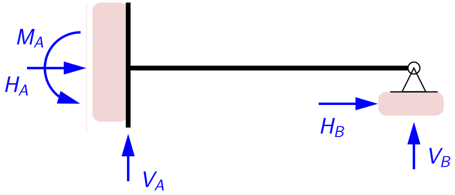
Example 1. Supported Cantilever Beam.

We have:

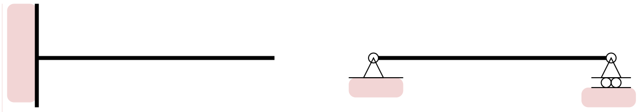
$r = 0$,

$f = \{V_A, H_A, \cdots, V_B\} =$
5.

$\hat{i} = f - 3 - r = 2.$

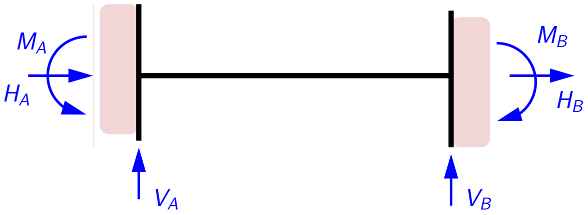


Need to release two restraints to create determinate structures, e.g.,



Indeterminacy of Beams

Example 2. Fixed-Fixed Beam.



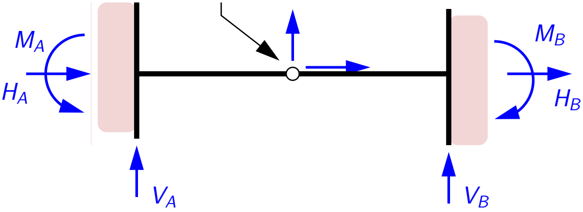
We have: $r = 0$,

$f = \{V_A, H_A, M_A, V_B, H_B, M_B\} = 6$.

$\hat{i} = f - 3 - r = 3$.

Indeterminacy of Beams

Example 3. Fixed-Fixed Beam + Hinge.



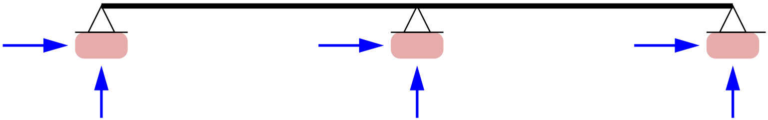
We have: $r = 1$,

$f = \{V_A, H_A, M_A, V_B, H_B, M_B\} = 6.$

$\hat{i} = \text{f - 3 - r} = 2.$

Indeterminacy of Beams

Example 4. Two-Span Beam.



We have: $r = 0$,

$f = \{V_A, H_A, V_B, H_B, V_C, H_C\} = 6$.

$\hat{i} =$ f - 3 - r $= 3$.

Indeterminacy of Frames

Tree Method

Approach: Systematically release redundant forces until trees are formed.

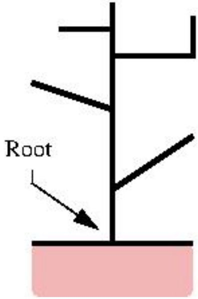
Formula: $\hat{i}$ = f - 3t, where:

- f = no of external forces,
- t = no of trees.

Constraints: Frame cannot have internal releases (no loops in trees).

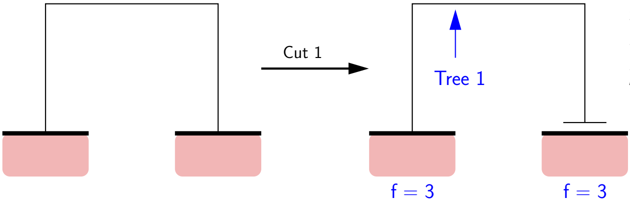
Trees:

- A tree has one root.
- A tree cannot have a closed loop branch.



Tree Method

Example 1a.

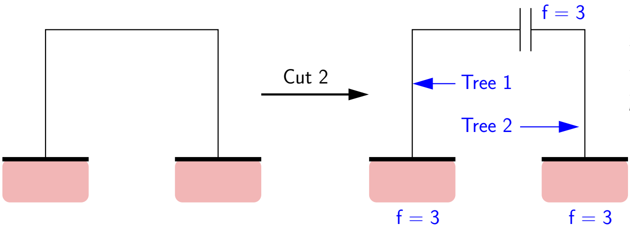


$f = 6$

$t = 1$

$\hat{i} = 6 - 3 = 3$

Example 1b.



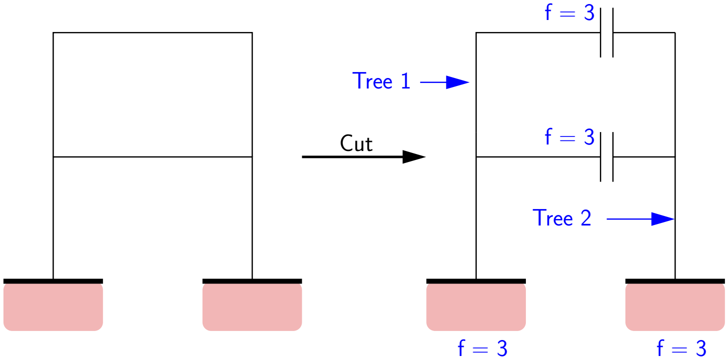
$f = 9.$

$t = 2.$

$\hat{i} = 9 - 3 * 2 = 3$

Tree Method

Example 2.



$f = 12.$

$t = 2.$

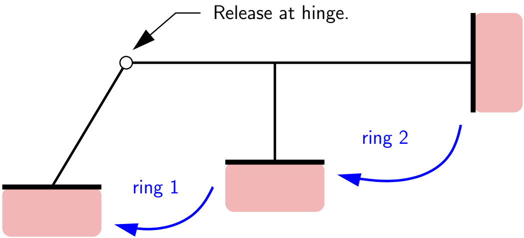
$\hat{i} = 12 - 3 * 2 = 6.$

Ring Method

Formula: $\hat{i}$ = 3n - r, where:

- n = no of rings.
- r = no of releases (each ring has 3 degrees of indeterminacy).

Example 1.



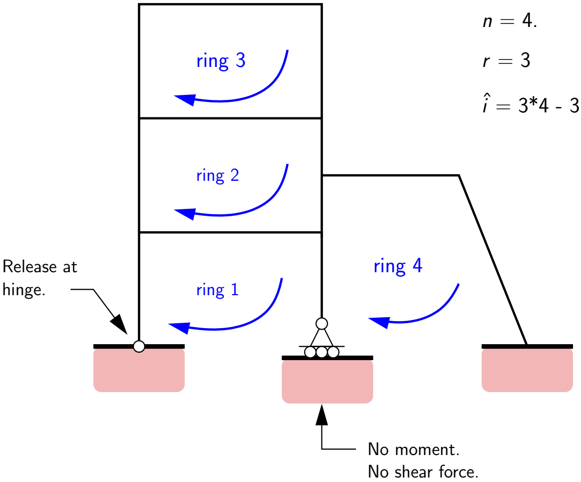
$n = 2.$

$r = 1.$

$\hat{i} = 6 - 1 = 5.$

Ring Method

Example 2.



$n = 4.$

$r = 3$

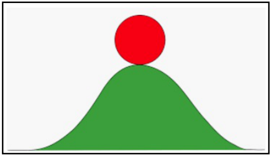
$\hat{i} = 3*4 - 3 = 9.$

Stability

Stability

Stability of Dynamical Systems

Instability occurs when some of the system outputs (e.g., displacement) can increase without bounds. In structural analysis, equilibrium of displacements corresponds to a minimum energy state.

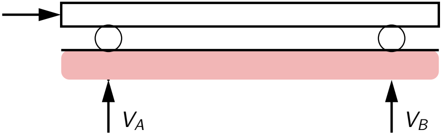


Instability of Structures:

- **External Instability.** When support reactions are either concurrent forces about a point or parallel.
- **Internal Instability.** When a mechanism exists.

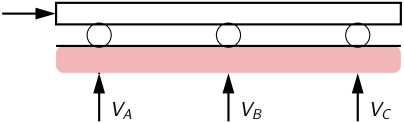
External Instability

Example 1: Reaction forces are parallel.



Three equations of equilibrium but only two reactions (r < 3n).

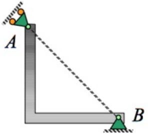
Example 2: Three parallel reaction forces

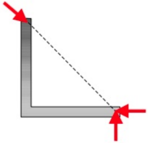


Three equations of equilibrium and three reactions (r = 3n). Still unstable.

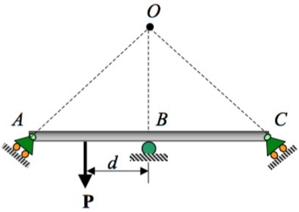
External Instability

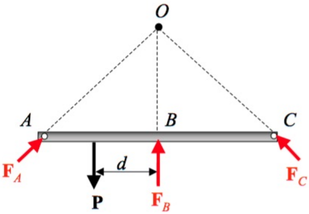
Example 3. Reaction forces are concurrent about point B.





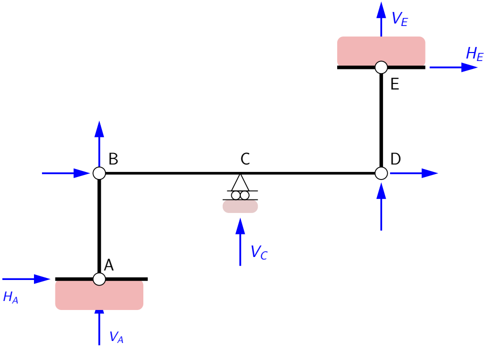
Example 4. Reaction forces are concurrent.





Internal Instability

Example 5. Structural configuration forms a pendulum mechanism.

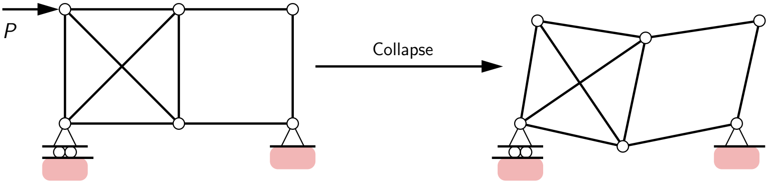


Three members: $n = 3$. No reactions r $= \{H_A, V_A, \cdots, H_E\} = 9$.

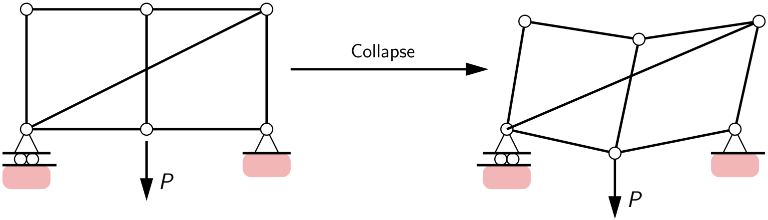
Test: r - 3n $= 0 \rightarrow$ statically determinate.

Internal Instability

Example 6: Internal Mechanism.



Example 7: Sometime internal mechanism are hard to identify.



Relating Stability to Linear Matrix Equations

Aside: If we compute the reactions and then systematically write the equations of equilibrium for each joint $\rightarrow 2j$ equations, which can be put in matrix form:

$$[A]\,[X] = [B]\,. \tag{2}$$

Here,

- X is a vector of truss element forces.
- A is a matrix of geometry and boundary conditions.
- B is a vector of applied loads.

When the system is statically determinate we can write:

$$[A]\,[X] = [B] \rightarrow [X] = \left[A^{-1}\right][B]\,. \tag{3}$$

Relating Stability to Linear Matrix Equations

Equations 3 only exist when $\left[A^{-1}\right]$ exists.

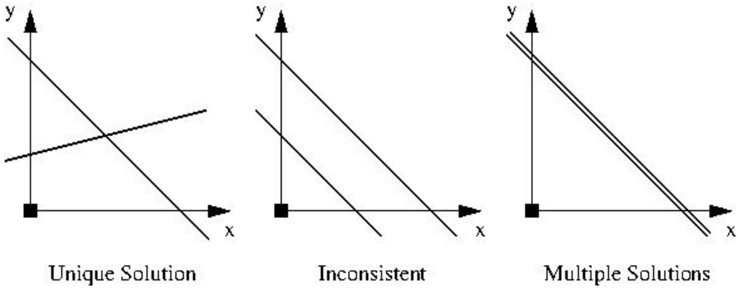
And this requires that the individual equations be linearly independent.

In Example 2, $\left[A^{-1}\right]$ does not exist because the reactions are co-linear, meaning that V_A can be written as a linear combination of V_B and V_C, i.e.,

$$\sum F_y = 0 \rightarrow V_A + V_B + V_C = 0. \tag{4}$$

Equations in Two Dimensions

Three Types of Solutions:



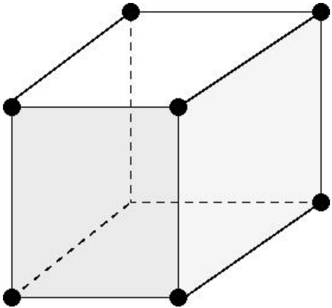
- Unique solution when two lines meet at a point.
- No solutions when two lines are parallel but not overlapping.
- Multiple solutions when two lines are parallel and overlap.

Equations in Three Dimensions

Also Three Types of Solutions:

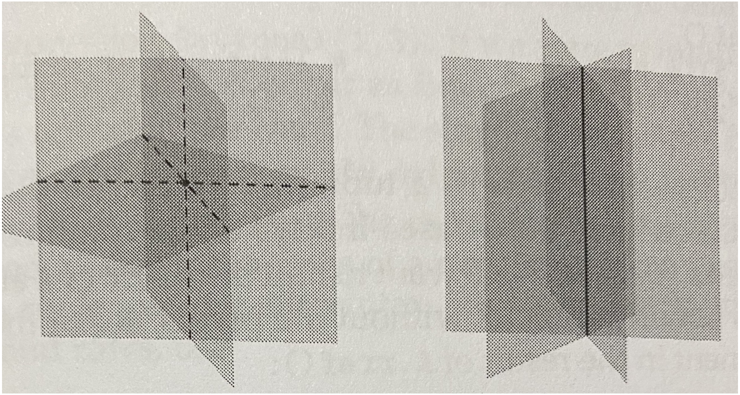
Each equation corresponds to a plane in three dimensions (e.g., think walls, floor and ceiling in a room).

- Unique solution when three planes intersect at a corner point.
- Multiple solutions where three planes overlap or meet along a common line.
- No solutions when three planes are parallel, but distinct, or pairs of planes that intersect along a line (or lines).



Equations in Three Dimensions

One Solution/Infinite Solutions:



Equations in Three Dimensions

No Solutions: