Department of Civil and Environmental Engineering, Spring Semester, 2020

ENCE 353 Final Exam, Open Notes and Open Book

Name : AUSTIN.

Exam Format and Grading. This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

Answer Question 1. Then answer **three of the five** remaining questions. Partial credit will be given for partially correct answers, so please show all your working.

IMPORTANT: Only the first four questions that you answer will be graded, so please **cross out the two questions you do not want graded** in the table below.

After you have finished working on the exam, look at the bonus problem for additional credit. No partial credit for this part of the exam.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
Bonus	4	
Total	50	

Question 1: 20 points

COMPULSORY: Analysis of a Cantilever with Moment Area and Method of Virtual Forces. Consider the cantilever shown in Figure 1.

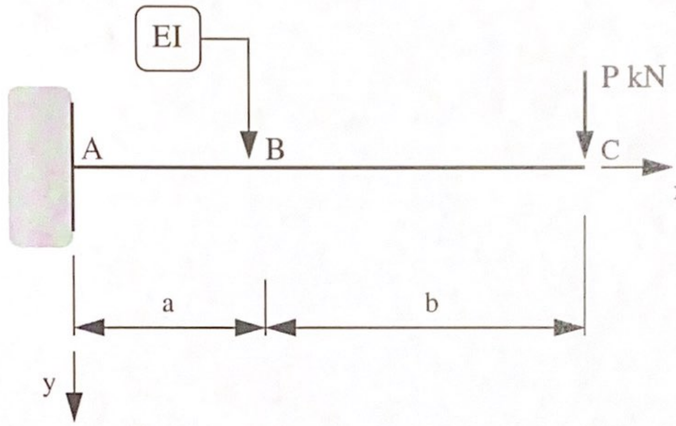


Figure 1: Front elevation view of a cantilever.

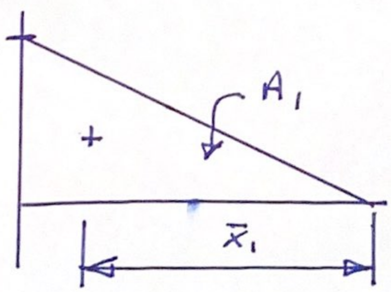
The cantilever has constant section properties, EI, along its entire length (a+b). A vertical load P (kN) is applied at point C.

[1a] (3 pts) Use the method of **MOMENT AREA** to show that the vertical deflection of the cantilever at point C is:

$$y_C = \frac{P(a+b)^3}{3EI}. \tag{1}$$

M/EI

$\frac{P(a+b)}{EI}$



$$A_1 = \frac{1}{2} \frac{P(a+b)}{EI} (a+b)$$

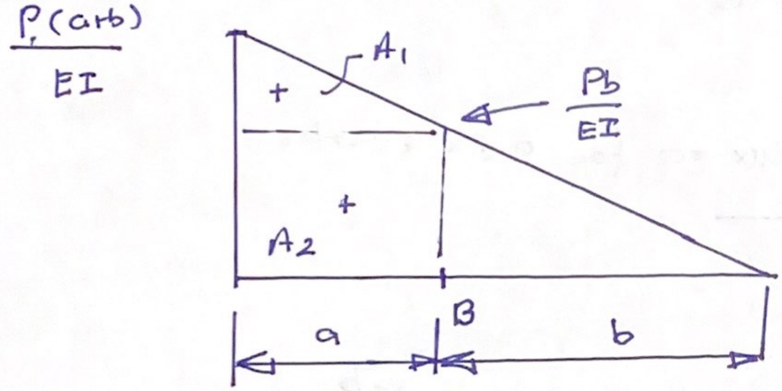
$$= \frac{1}{2} \frac{P(a+b)^2}{EI}.$$

$$\bar{x}_1 = \frac{2}{3} (a+b).$$

$$y_c = A_1 \bar{x}_1 = \frac{1}{3} \frac{P(a+b)^3}{EI}.$$

[1b] (3 pts) Use the method of **MOMENT AREA** to show that the vertical deflection of the cantilever at point B is:

$$y_B = \frac{Pa^2}{6EI}[3b + 2a]. \tag{2}$$



$$A_1 = \frac{1}{2}\left(\frac{P(a+b)}{EI} - \frac{Pb}{EI}\right)a = \frac{1}{2}\frac{Pa^2}{EI}.$$

$$A_2 = \frac{Pb}{EI}\cdot a.$$

$$\bar{x}_1 = \tfrac{2}{3}a, \quad \bar{x}_2 = \tfrac{a}{2}$$

$$\Rightarrow \quad y_B = A_1\bar{x}_1 + A_2\bar{x}_2$$

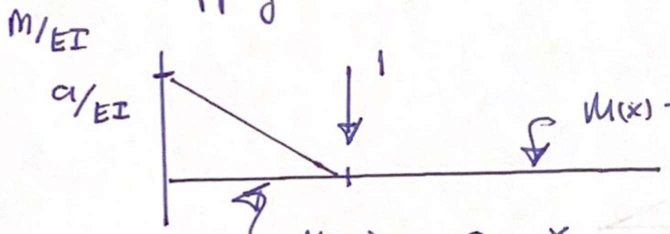
$$= \frac{Pa^2}{6EI}(3b + 2a).$$

[1c] (3 pts) Use the method of **VIRTUAL FORCES** to show that the vertical deflection of the cantilever at point B is:

$$y_B = \frac{Pa^2}{6EI}[3b + 2a]. \tag{3}$$

Hint: You can simplify the integration by adopting a transformation $z = a - x$.

Apply unit load at B.



$M(x) = 0$ for $a \leq x \leq a+b$.

$$\frac{M(x)}{EI} = \frac{a-x}{EI}.$$

$$y_B = \frac{1}{EI}\int_0^a \underbrace{P(a+b-x)}_{M(x).}\underbrace{(a-x)}_{m(x)}\,dx + \int_a^{a+b} 0\ldots\,dx$$

$$= \frac{Pa^2}{6EI}(3b+2a).$$

Now suppose that a roller support is inserted below point B as follows:

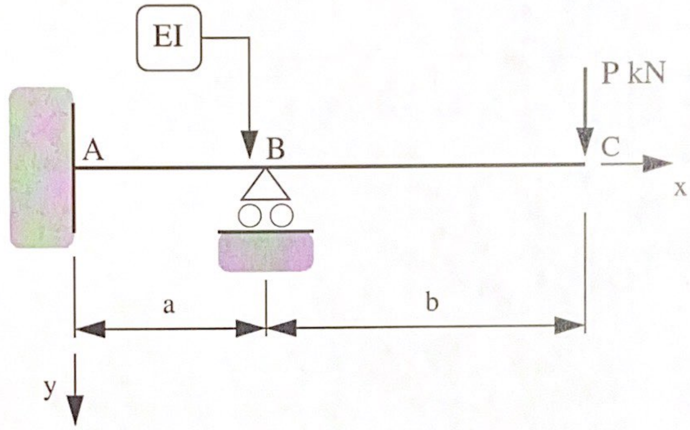
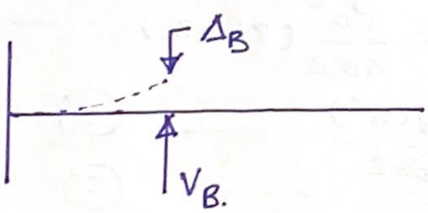


Figure 2: Front elevation view of a cantilever supported by a roller at point B.

[1d] (3 pts) Show that the vertical support reaction at B is:



$$V_b = \frac{P}{2}\left[\frac{3b + 2a}{a}\right]. \tag{4}$$

Δ_B = deflection at B due to V_B

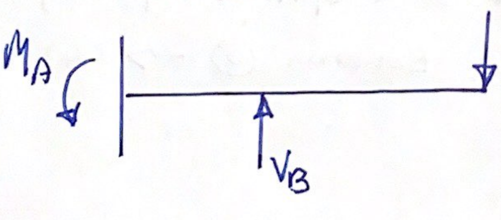
$= \frac{1}{3} \frac{V_B a^3}{EI}.$

Net vertical deflection at B = 0

$\Rightarrow \frac{1}{3} \frac{V_B a^3}{EI} = \frac{Pa^2}{6EI}(3b + 2a)$

$\Rightarrow V_B = \frac{P}{2}\left[\frac{3b + 2a}{a}\right]$

[1e] (3 pts) Hence, derive a simple expression for the bending moment at A.



$\sum M_A = 0$

$\Rightarrow M_A = P(a+b) - V_B . a$

$= -\frac{Pb}{2}.$

Finally, let's replace the roller support below point B with a spring.

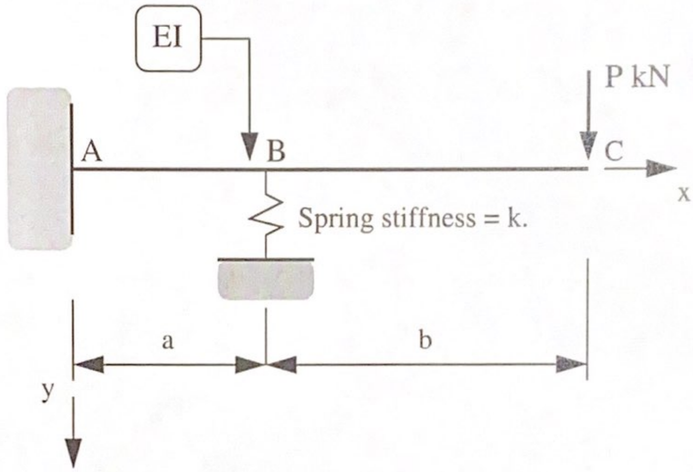


Figure 3: Cantilever supported by a spring at point B.

[1f] (3 pts) Show that the support reaction, V_b, is now given by the equation:

$$V_b\left[\frac{1}{k}+\frac{a^3}{3EI}\right]=\frac{Pa^2}{6EI}[3b+2a]. \tag{5}$$

Deflection at B due to P at C $\Rightarrow \Delta_{B1} = \frac{Pa^2}{6EI}(3b+2a)$ — (A)

Deflection at B due to V_B $\Rightarrow \Delta_V = \frac{V_B(a^3)}{3EI}$ —— (B)

Spring behaviour: $k(\Delta_B - \Delta_V) = V_B$ —— (C)

Plug (A) & (B) into (C): $\frac{Pa^2}{6EI}(3b+2a) - \frac{V_B a^3}{3EI} = \frac{V_B}{K}$.

$\Rightarrow V_B\left[1/k + \frac{a^3}{3EI}\right] = \frac{Pa^2}{6EI}(3b+2a)$. — (D)

[1g] (2 pts) Explain why V_b for spring support (i.e., equation 5) is always lower than for roller support (i.e., equation 4).

The roller support is equivalent to a spring support, where the spring has infinite stiffness. Equation (D) $\rightarrow$ (4) from below as $k \rightarrow \infty$.

Question 2: 10 points

OPTIONAL: Moment-Area Method. Consider the cantilevered beam structure shown in Figure 4.

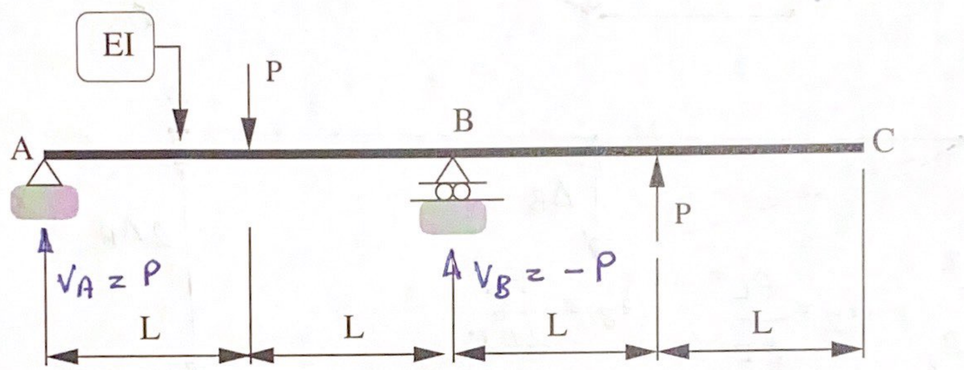
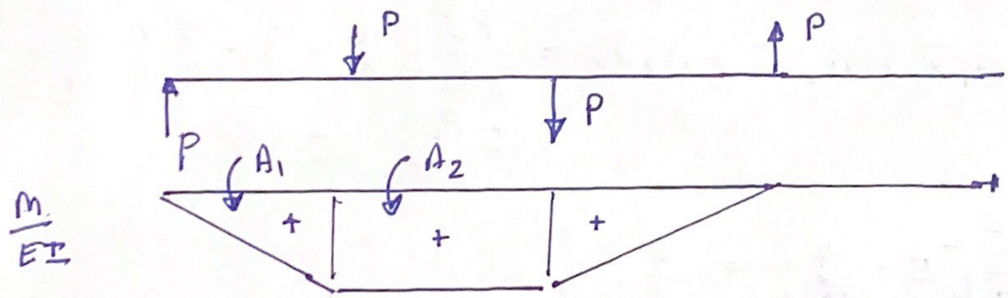


Figure 4: Front elevation view of a cantilevered beam structure.

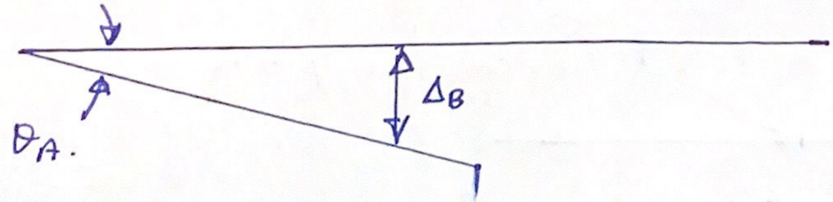
[2a] (4 pts) Use the method of **MOMENT AREA** to compute the rotation at point A.



$A_1 = \frac{PL^2}{2EI}$

$A_2 = \frac{PL^2}{EI}$.

Moment Area.



$\bar{x}_1 = 4/3 L$,

$\bar{x}_2 = L/2$.

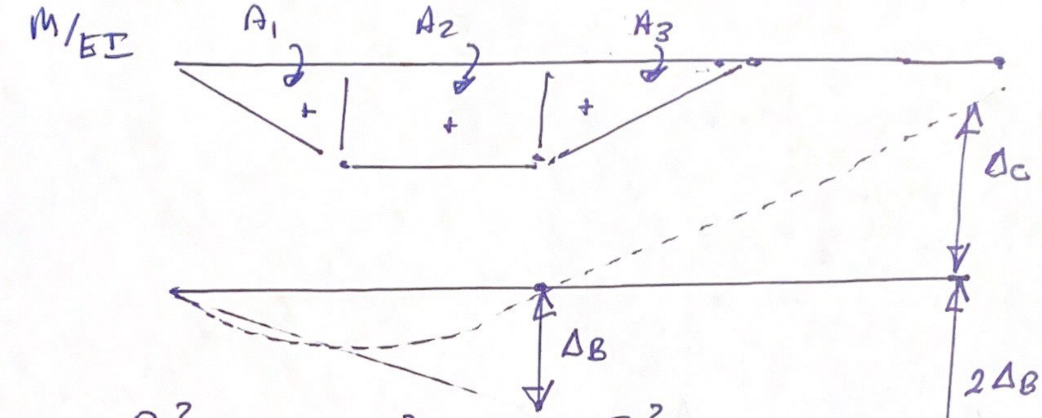
$$\Delta_B = A_1 \bar{x}_1 + A_2 \bar{x}_2 = \frac{PL^2}{2EI} \cdot \frac{4}{3} L + \frac{PL^2}{EI} \cdot \frac{L}{2} = \frac{PL^3}{EI} \cdot \frac{14}{12}$$

From geometry:

$$\theta_A = \Delta_B / 2L = \frac{7}{12} \frac{PL^2}{EI}.$$

— A

[2b] (4 pts) Use the method of **MOMENT AREA** to compute the vertical deflection of the beam at point C.



$$A_1 = \frac{PL^2}{2EI}, \quad A_2 = \frac{PL^2}{EI}, \quad A_3 = \frac{PL^2}{2EI}$$

$$\bar{x}_1 = \frac{10}{3}L, \quad \bar{x}_2 = \frac{5}{2}L, \quad \bar{x}_3 = \frac{5}{3}L.$$

Apply moment Area about point C:

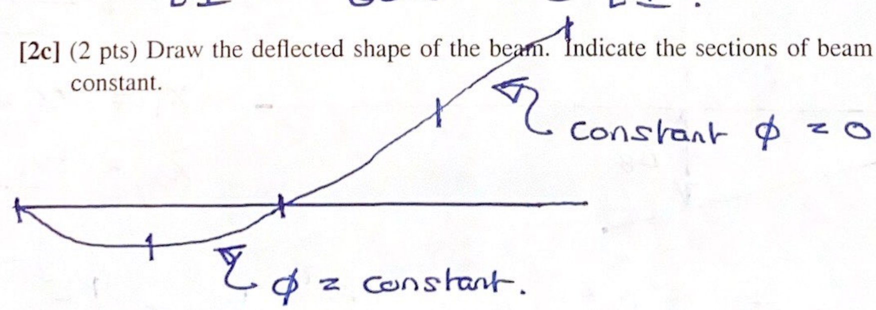
$$2\Delta_B + \Delta_C = A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 = \frac{PL^3}{EI}\left[\frac{10}{6} + \frac{5}{2} + \frac{5}{6}\right]$$

$$= 5\frac{PL^3}{EI}. \quad \text{— (B)}$$

Plug (A) -> (B):

$$\Delta_C = \frac{PL^3}{EI}\left[5 - \frac{14}{6}\right] = \frac{8}{3}\frac{PL^3}{EI}.$$

[2c] (2 pts) Draw the deflected shape of the beam. Indicate the sections of beam where the curvature is constant.



Question 3: 10 points

OPTIONAL: Structural Analysis of a Simple Beam Structure. The beam structure shown in Figure 5 supports a uniformly distributed load w (N/m) between points B and C.

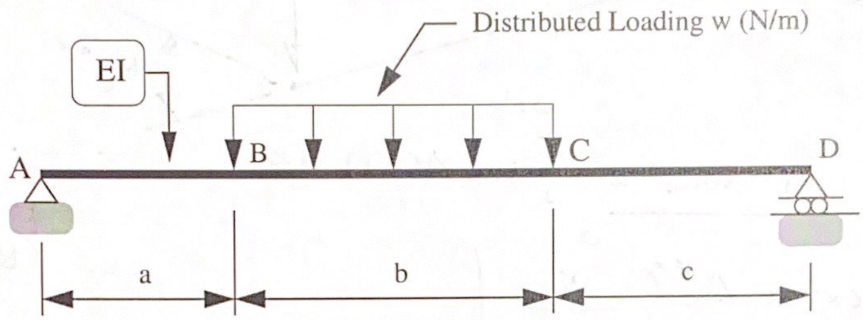
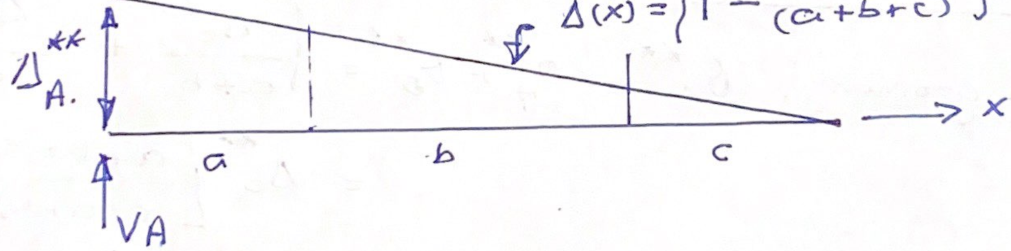


Figure 5: Front elevation view of a simple beam structure.

[3a] (4 pts) Use the method of **VIRTUAL DISPLACEMENTS** to compute formulae for the vertical reactions at A and D. Show all of your working.

Apply virtual displacement Δ^{**} at A.

$$\Delta(x) = \left\{ 1 - \frac{x}{(a+b+c)} \right\} \Delta_A^{**}.$$



Energy Balance: $\sum' EWD = 0$

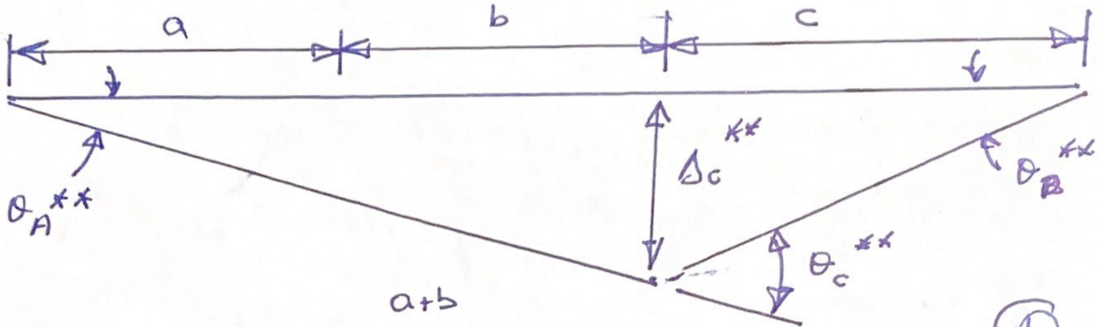
$$\Rightarrow V_A \Delta_A^{**} - \int_a^{a+b} w \, \Delta(x) \, dx = 0.$$

$$\Rightarrow V_A = \frac{wb(b+2c)}{2(a+b+c)}.$$

Similarly for V_D $\Rightarrow V_D = \dfrac{wb(2a+b)}{2(a+b+c)}.$

Note: $V_A + V_D = wb.$ ✓

[3b] (6 pts) Use the method of **VIRTUAL DISPLACEMENTS** to compute a formula for the bending moment at C. Show all of your working.



<u>External work done</u> $= \int_a^{a+b} w(\Delta(x))\,dx$ — (A)

but $\Delta(x) = \Delta_c^{**}\left[\frac{x}{(a+b)}\right]$ — (B)

Plug (B) -> (A) and integrate.

$$EWD = w\Delta_c^{**}\left[\frac{x^2}{2(a+b)}\right]_a^{a+b} = \frac{w\Delta_c^{**}}{2(a+b)}\left[2ab+b^2\right]$$

<u>Internal Work done</u> $= M_c'\,\theta_c^{**}$

From geometry. $\theta_c^{**} = \theta_A^{**} + \theta_D^{**} = \left[\frac{\Delta_c^{**}}{a+b} + \frac{\Delta_c^{**}}{c}\right]$

$$= \Delta_c^{**}\left[\frac{a+b+c}{c(a+b)}\right] \quad -(c)$$

Equating EWD = IWD.

$$\Rightarrow M_c\,\theta_c^{**} = \frac{w\Delta_c^{**}}{2(a+b)}.\,(2ab+b^2)$$

$$\Rightarrow M_c = \frac{2\Delta_c^{**}}{2(a+b)}.\,(2ab+b^2).\,\frac{1}{\Delta_c^{**}}\left[\frac{c(a+b)}{a+b+c}\right]$$

$$= \frac{wbc\,(2a+b)}{2(a+b+c)}$$

<u>Note:</u> $M_c = V_D \cdot c$ ✓.

Question 5: 10 points

OPTIONAL: Principle of Virtual Work. Consider the supported cantilevered beam structure shown in Figure 7.

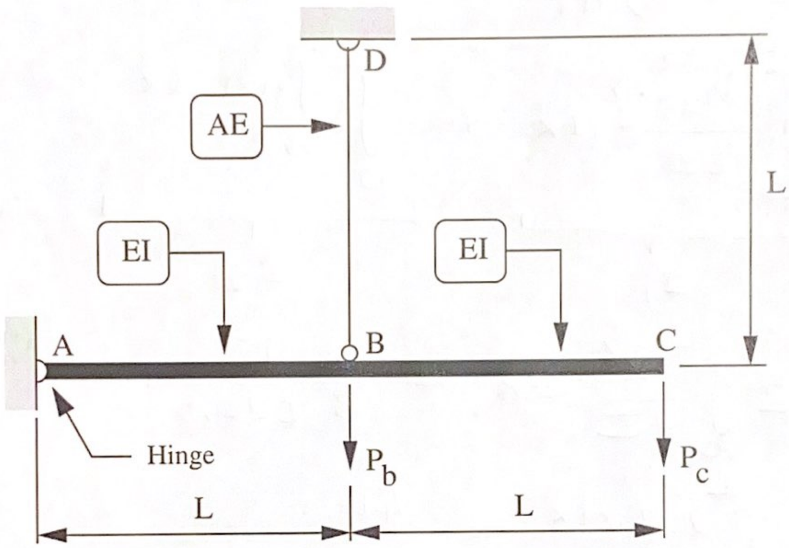
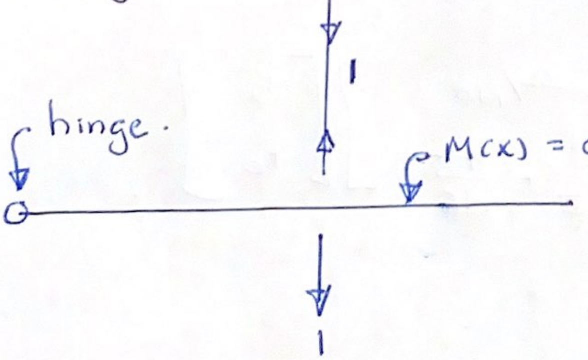


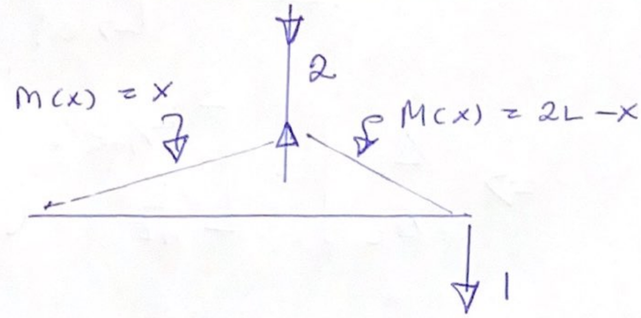
Figure 7: Front elevation view of a supported cantilevered beam structure.

Use the principle of **VIRTUAL FORCES** to compute the two-by-two flexibility matrix connecting displacements at points B and C to applied loads P_b and P_c, i.e.,

$$\begin{bmatrix} \triangle_b \\ \triangle_c \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_b \\ P_c \end{bmatrix}. \tag{7}$$

① Apply unit force at B. ② Apply unit force at C.





Question 5 continued ...

$$f_{11} = \underbrace{\int_0^{2L} \frac{M_1^2(x)}{EI}\,dx}_{\hookrightarrow 0!} + \int_0^{L} \frac{f_1^2(x)\,dx}{AE} = \boxed{\frac{L}{AE}}$$

$$f_{22} = \underbrace{\int_0^{2L} \frac{M_2^2(x)}{EI}\,dx}_{I_1} + \underbrace{\int_0^{L} \frac{f_2^2(x)\,dx}{AE}}_{I_2}$$

$$I_1 = 2\int_0^{L} \frac{x^2}{EI}\,dx = \frac{2}{3}\frac{L^3}{EI}.$$

$$I_2 = \int_0^{L} \frac{2.2\,dx}{AE} = \frac{4L}{AE}.$$

$$\Rightarrow f_{22} = \frac{2}{3}\frac{L^3}{EI} + \frac{4L}{AE}.$$

$$f_{12} = \underbrace{\int_0^{2L} \frac{M_1(x).\,M_2(x)\,dx}{EI}}_{\hookrightarrow 0!} + \int_0^{L} \frac{f_1(x).\,f_2(x)}{AE}\,dx = \frac{2L}{AE}.$$

$$\Rightarrow \begin{bmatrix} \Delta_b \\ \Delta_c \end{bmatrix} = \begin{bmatrix} L/AE & 2L/AE \\ \frac{2L}{AE} & \left(\frac{2}{3}\frac{L^3}{EI} + \frac{4L}{AE}\right) \end{bmatrix} \begin{bmatrix} P_b \\ P_c \end{bmatrix}.$$

Question 6: 10 points

OPTIONAL: Compute Support Reactions in a Suspension Bridge. Figure 8 is a front elevation view of a suspension bridge that has three spans, two support towers, and anchor supports at the bridge endpoints.

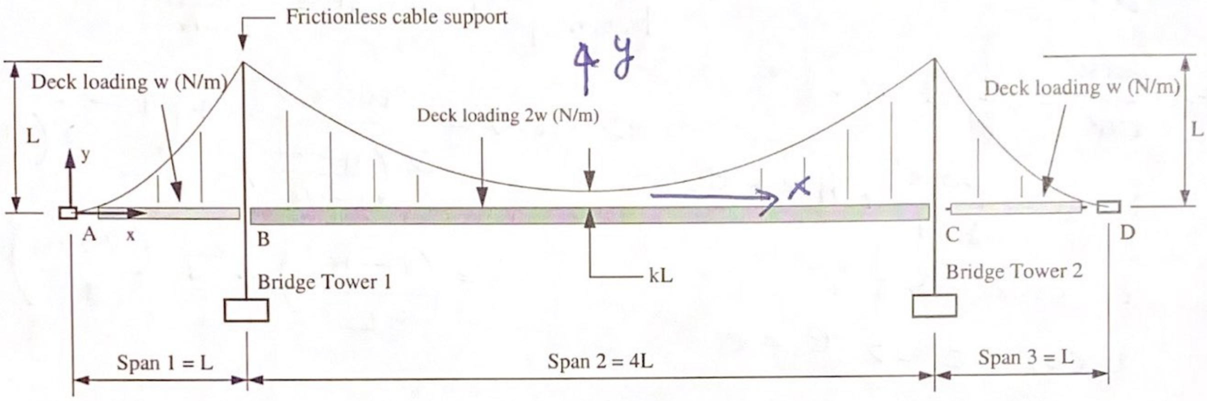


Figure 8: Elevation view of a three-span suspension bridge.

Spans 1, 2 and 3 have lengths L, $4L$ and L, respectively. Spans 1 and 3 carry a uniform load w (N/m). Span 2 carries a uniform load $2w$ (N/m). The towers have height L above the bridge deck level. At the mid-point of Span 2, the lowest point of the cable profile is kL above the bridge deck.

The purpose of this question is to work step by step toward the computation of vertical support reactions at the cable anchor points (i.e., at points A and D), and at the towers (i.e., at points B and C). For the purposes of this analysis, assume that the bridge cable weight is bundled into the applied loads w, and can otherwise be ignored. Also, assume that the cable passes through the top of the towers on a frictionless support and, as a result, the horizontal component of cable force will be constant along the entire bridge.

[6a] (2 pts) By examining equilibrium of the cable profile in Span 2, show that the horizontal component of cable force is:

$$H = \left[\frac{4}{1-k}\right] wL. \tag{8}$$

Set origin at middle of Span 2.

$$\frac{d^2y}{dx^2} = \frac{2w}{H} \rightarrow \frac{dy}{dx} = \frac{2wx}{H} + A; \quad \left.\frac{dy}{dx}\right|_{x=0} = 0 \Rightarrow A = 0.$$

$$y(x) = \frac{wx^2}{2H} + B: \quad y(0) = 0 \Rightarrow B = 0.$$

$$y(2L) = L(1-k) \Rightarrow H = \left\{\frac{4wL}{1-k}\right].$$

[6b] (3 pts) Show that the equation of the cable profile in Span 1 is:

$$y(x) = \left[\frac{1-k}{8L}\right]x^2 + \left[\frac{7+k}{8}\right]x \tag{9}$$

$$\frac{d^2y}{dx^2} = \frac{w(x)}{H} = \frac{w}{H}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{wx}{H} + A.$$

$$y(x) = \frac{wx^2}{2H} + Ax + B.$$

Boundary conditions:

$$y(0) = 0 \Rightarrow B = 0$$

$$y(L) = L \Rightarrow$$

$$L = \frac{wL^2}{2H} + AL.$$

$$\Rightarrow A = 1 - \frac{wL}{2H}.$$

From part (a) $H = \dfrac{4wL}{1-k}$

$$\Rightarrow y(x) = \frac{w(x^2)}{2\left(\frac{4wL}{1-k}\right)} + \left(1 - \frac{wL}{2\left(\frac{4wL}{1-k}\right)}\right)x$$

$$= \left(\frac{1-k}{8L}\right)x^2 + \left(\frac{7+k}{8}\right)x.$$

[6c] (3 pts) Use the results from parts [6a] and [6b] to show that the vertical reaction at the anchor support (i.e., at point A) is:

$$V_A = \left[\frac{7+k}{2-2k}\right]wL \tag{10}$$

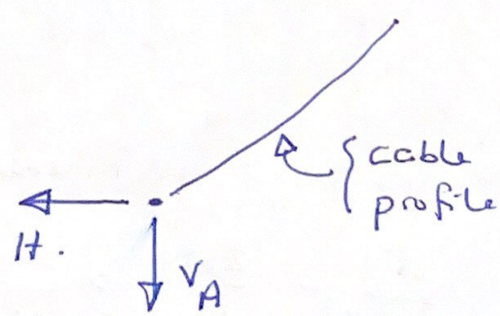
acting downwards.

Cable geometry:

$$\tan(\theta) = \frac{dy}{dx} = \frac{V}{H}.$$

At $x = 0$

$$\left.\frac{dy}{dx}\right|_{x=0} = \left(1 - \frac{wL}{2H}\right) = \frac{V_A}{H}$$



$$\Rightarrow V_A = \frac{7+k}{2(1-k)} wL.$$

[6d] (2 pts) Hence, show that the vertical support reaction at Tower 1 is:

$$V_B = \left[\frac{17 - 9k}{2 - 2k}\right] wL. \tag{11}$$

acting upwards.

Total downward deck loading = $10wL$.

Using symmetry, V_A & V_B carry $5wL$

& $V_C + V_D$ carry $5wL$.

Left-hand side:

$$\sum' F_y = 0 \Rightarrow V_A + V_B = 5wL.$$

notice that V_A acts downwards

$$\Rightarrow V_B = V_A + 5wL = \left(\frac{7+k}{2-2k}\right) wL + \left\{\frac{10-10k}{(2-2k)}\right\} wL$$

$$= \left[\frac{17 - 9k}{2-2k}\right] wL.$$