Department of Civil and Environmental Engineering, Fall Semester, 2020

ENCE 353 Midterm 2, Open Notes and Open Book

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E-mail (print neatly!):

Exam Format and Grading. Attempt all three questions. Partial credit will be given for partially correct answers, so please **show all of your working**.

Question	Points	Score
1	15	
2	15	
3	10	
Total	40	

Question 1: 15 points

Analysis of a Cantilever Beam with Moment Area. Consider the cantilever shown in Figure 1.

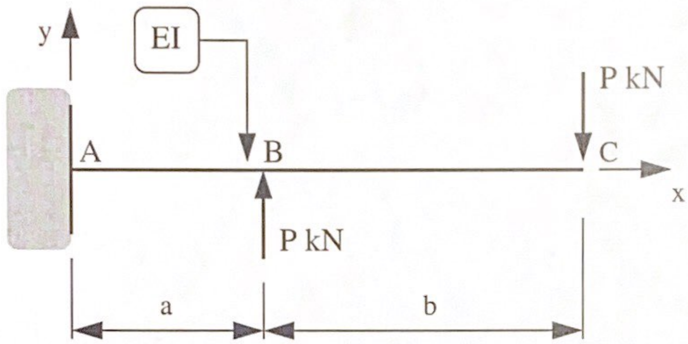
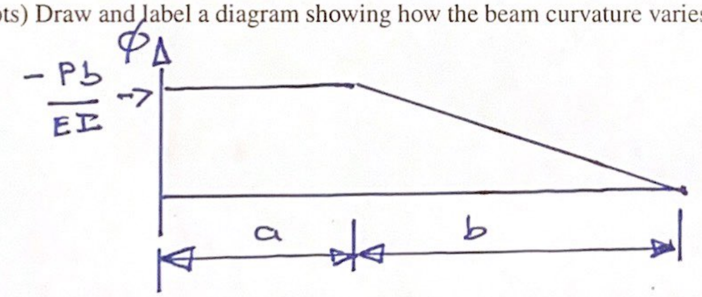


Figure 1: Front elevation view of a cantilever.

The cantilever has constant section properties, EI, along its entire length (a+b). Vertical loads of P (kN) are applied at points B and C. Notice that the y axis is pointing upwards – hence, if we apply the right-hand rule, positive rotations will be anti-clockwise.

[1a] (3 pts) Draw and label a diagram showing how the beam curvature varies along its length (i.e., $\phi(x)$).



$$\phi(x) = \left[M(x)/EI \right]$$

$$M(x) = \begin{cases} -Pb, & 0 < x \leq a \\ -P(a+b-x), & (a < x \leq a+b) \end{cases}$$

[1b] (3 pts) Use the method of **moment-area** to show that the rotation of point B is:

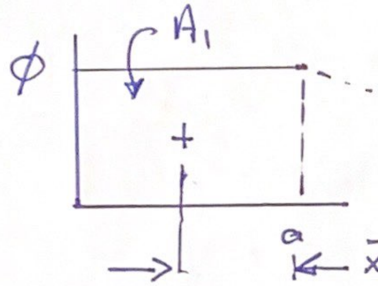
$$\theta(a) = \left[\frac{-Pba}{EI} \right]. \tag{1}$$

$\theta(a)$ = area of M/EI diagram between $x = 0$ & $x = a$

$$= \frac{-Pba}{EI}.$$

[1c] (4 pts) Use the method of **moment-area** to show that the vertical displacement of point B is:

$$y(a) = \left[\frac{-Pba^2}{2EI}\right]. \tag{2}$$

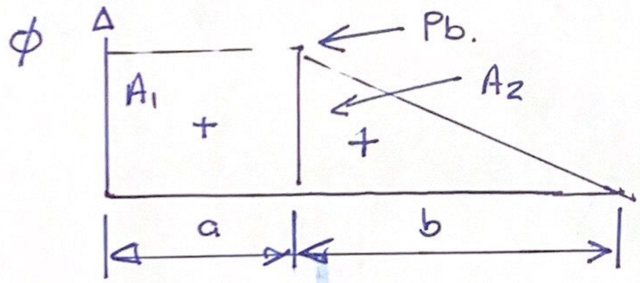


$$\left.\begin{aligned} A_1 &= -\frac{Pba}{EI} \\ x_1 &= a/2. \end{aligned}\right\} \quad y(a) = A_1 . \bar{x}_1 = -\frac{Pba^2}{2EI}.$$

[1d] (5 pts) Use the method of **moment-area** to show that the vertical displacement of point C is:

$$y(a+b) = \left[\frac{-Pb}{6EI}\right]\left[6ab + 3a^2 + 2b^2\right]. \tag{3}$$

Note: Notice that when a = 0, equation 3 simplifies to the formula we have seen many times in class.



$$A_1 = \frac{Pba}{EI}$$

$$A_2 = \frac{1}{2}\frac{Pb^2}{EI}$$

$$\bar{x}_1 = b + a/2$$

$$\bar{x}_2 = 2/3\, b.$$

$$\begin{aligned} y(a+b) &= -\left[A_1 \bar{x}_1 + A_2 \bar{x}_2\right] \\ &= \frac{-Pba}{EI}(b + a/2) + \frac{-Pb^2}{2EI}\left(2/3\, b\right) \\ &= \frac{-Pb}{6EI}\left[6ab + 3a^2 + 2b^2\right]. \end{aligned}$$

Question 2: 15 points

Elastic Curve for a Cantilever Beam Structure. Consider the cantilever shown in Figure 2.

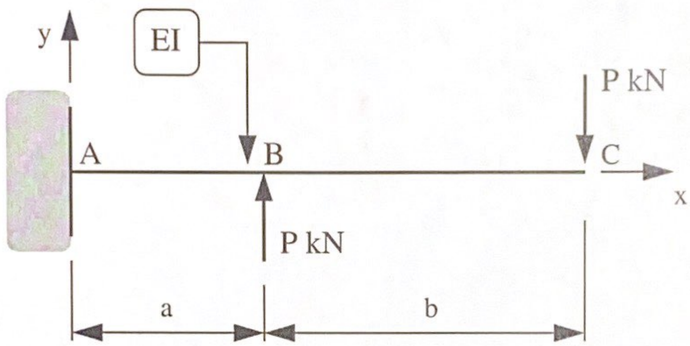


Figure 2: Front elevation view of a cantilever.

The cantilever has constant section properties, EI, along its entire length (a+b). Vertical loads of P (kN) are applied at points B and C.

[2a] (3 pts) Write a mathematical formula, M(x), for the bending moment along the beam as a function of x.

$$M(x) = \begin{cases} -Pb, & 0 < x \leq a \\ -P(a+b-x), & a \leq x \leq a+b \end{cases}$$

[2b] (4 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{M(x)}{EI}\right], \tag{4}$$

and appropriate boundary conditions, show that in the interval $0 \le x \le a$, the beam displacement and rotation are:

$$y(x) = \left[\frac{-Pbx^2}{2EI}\right] \quad \text{and} \quad \theta(x) = \left[\frac{-Pbx}{EI}\right]. \tag{5}$$

Show all of your working.

Within the interval $0 \le x \le a$,

$$\frac{d^2y}{dx^2} = \frac{-Pb}{EI}.$$

Integrating twice gives:

$$\frac{dy}{dx} = \frac{-Pbx}{EI} + A$$

$$y(x) = \frac{-Pbx^2}{2EI} + Ax + B.$$

Boundary conditions:

$$y(0) = \left.\frac{dy}{dx}\right|_{x=0} = 0$$

$$\Rightarrow A = 0 \ \& \ B = 0.$$

Thus, $y(x) = \dfrac{-Pbx^2}{2EI}$ and $\dfrac{dy}{dx} = \theta(x) = \dfrac{-Pbx}{EI}$.

[2c] (7 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{M(x)}{EI}\right], \tag{6}$$

and appropriate boundary conditions, show that within the interval $a \leq x \leq a+b$, the beam displacement is:

$$y(x) = \left[\frac{-P}{6EI}\right]\left[3(a+b)x^2 - x^3 - 3a^2x + a^3\right]. \tag{7}$$

Hint: The algebra for this question is a bit tricky so I suggest that you work out the details on paper and submit a tidy summary of your procedure. Also notice that equation 7 is consistent with equations 2 and 3 in Question 1.

Boundary conditions.

At $x = a$, $y(a) = \dfrac{-Pba^2}{2EI}$ & $\dfrac{dy}{dx} = \dfrac{-Pba}{EI}$.

At $x = a+b$, $\left.\dfrac{d^2y}{dx^2}\right|_{x=a+b} = 0$ ← because $M(a+b) = 0$.

Within our interval of interest, $M(x) = -P(a+b-x)$.

Hence, from equation 6,

$$-\left[\frac{EI}{P}\right]\frac{d^2y}{dx^2} = a+b-x. \quad \text{— } (A)$$

Integrating (A) twice.

$$-\left(\frac{EI}{P}\right)\frac{dy}{dx} = (a+b)x - \frac{x^2}{2} + A. \quad \text{— } (B)$$

$$-\left(\frac{EI}{P}\right)y(x) = \frac{(a+b)x^2}{2} - \frac{x^3}{6} + Ax + B. \quad \text{— } (C)$$

Question 2c continued:

At $x = a$.

$$\left(-\frac{EI}{P}\right)\frac{dy}{dx} = \underbrace{ba}_{\text{boundary condition}} = \underbrace{(a+b)a - \frac{a^2}{2} + A}_{\text{Equation (B)}}$$

$$\Rightarrow A = -a^2/2.$$

$$\left(-\frac{EI}{P}\right)y(a) = \underbrace{\frac{ba^2}{2}}_{\text{boundary condition}} = \underbrace{\frac{(a+b)a^2}{2} - \frac{a^3}{6} - \frac{a^2}{2}.a + B}_{\text{Equation (C)}}.$$

$$\Rightarrow \cancel{\frac{ba^2}{2}} = \cancel{\frac{ba^2}{2}} + \cancel{\frac{a^3}{2}} - \frac{a^3}{6} - \cancel{\frac{a^3}{2}} + B.$$

$$\Rightarrow B = a^3/6.$$

Collecting terms:

$$y(x) = \frac{-P}{6EI}\left[3(a+b)x^2 - x^3 - 3a^2x + a^3\right] \quad - (D)$$

Matches moment area ✓.

Notice: $y(a) = \dfrac{-Pba^2}{2EI}$ ← Matches moment area ✓.

$$y(a+b) = \frac{-P}{6EI}\left[3(a+b)^3 - (a+b)^3 - 3a^2(a+b) + a^3\right]$$

$$= \frac{-P}{6EI}\left[6ab + 3a^2 + 2b^2\right]$$ ← Also matches moment area ✓.

Question 3: 10 points

Simple Three-Pinned Arch. Figure 3 is a front elevation view of a simple three-pinned arch that carries a total snow loading of 3WL uniformly distributed over its upper section and a point loading P (kN) applied at point C.

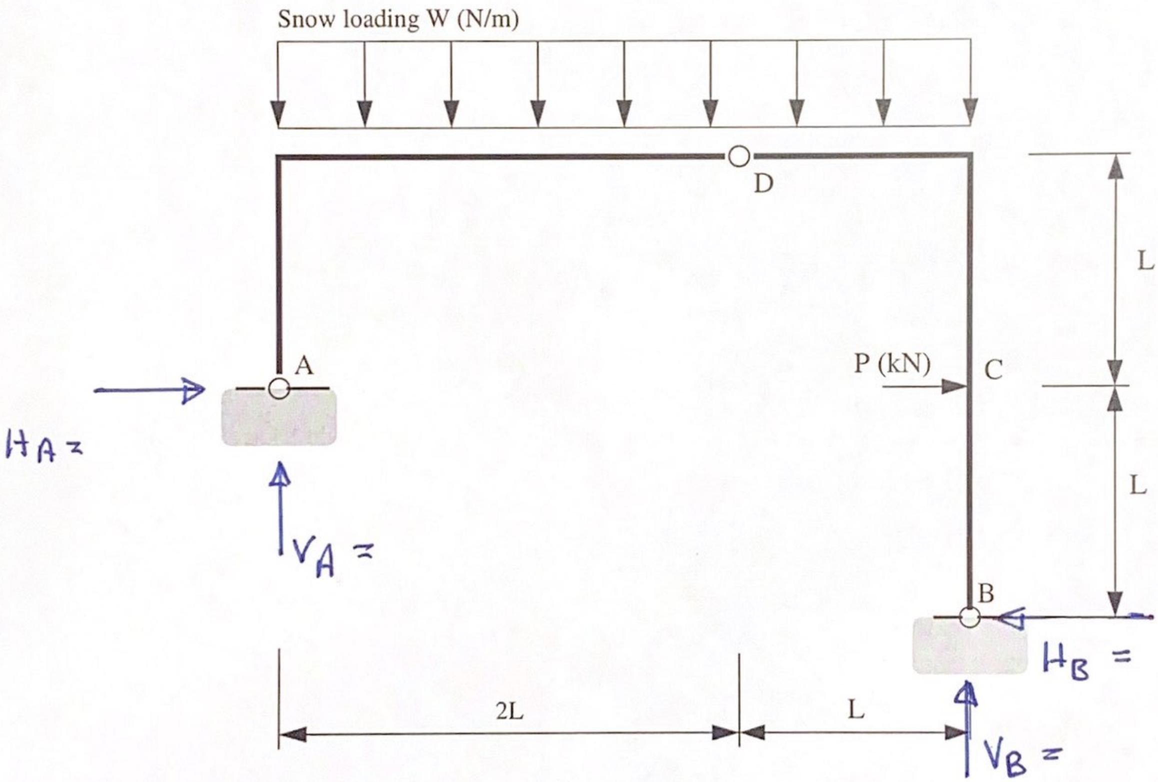


Figure 3: Front elevation view of a three-pinned arch that supports a snow loading.

[3a] (6 pts) Compute the vertical and horizontal components of reaction force at supports A and B as a function of W, L and P.

$\sum V = 0 \Rightarrow V_A + V_B = 3WL.$ —— (A)

$\sum H = 0 \Rightarrow H_A + P = H_B.$ —— (B)

$\sum M_D = 0$ (left-hand substructure).

$(2WL)L + H_A L = 2LV_A.$

$\Rightarrow 2WL + H_A = 2V_A$ —— (C)

Queation 3a continued:

$$\sum M_D = 0 \text{ (RHS)}$$

$$(WL)\frac{L}{2} + H_B \cdot 2L - PL = V_B L$$

$$\Rightarrow WL + 4H_B - 2P = 2V_B \qquad \text{—— (D)}$$

Solving (A) - (D).

$$H_A = \tfrac{3}{5}WL - \tfrac{2}{5}P$$

$$H_B = \tfrac{3}{5}WL + \tfrac{3}{5}P$$

$$V_A = \frac{13\,WL}{10} - \tfrac{P}{5}$$

$$V_B = \frac{17\,WL}{10} + \tfrac{P}{5}$$

Can validate by substituting into (A), (B) & (D).

[3b] (4 pts) Draw and label the bending moment diagram. Lets draw BMD for WL & P loadings seperately.