## ENCE 353 Final Exam, Open Notes and Open Book

Name:

Exam Format and Grading. This take home final exam is open notes and open book. You need to comply with the university regulations for academic integrity.

Answer Question 1. Then answer three of the five remaining questions. Partial credit will be given for partially correct answers, so please show all your working.

IMPORTANT: Only the first four questions that you answer will be graded, so please cross out the two questions you do not want graded in the table below.

After you have finished working on the exam, look at the bonus problem for additional credit. No partial credit for this part of the exam.

| Question | Points | Score |
| :---: | :---: | :---: |
| 1 | 20 |  |
| 2 | 10 |  |
| 3 | 10 |  |
| 4 | 10 |  |
| 5 | 10 |  |
| 6 | 10 |  |
| Bonus | 4 |  |
| Total | 50 |  |

## Question 1: 20 points

COMPULSORY: Analysis of a Cantilever with Moment Area and Method of Virtual Forces. Consider the cantilever shown in Figure 1.


Figure 1: Front elevation view of a cantilever.

The cantilever has constant section properties, EI, along its entire length $(a+b)$. A vertical load $\mathrm{P}(\mathrm{kN})$ is applied at point C .
[1a] (3 pts) Use the method of MOMENT AREA to show that the vertical deflection of the cantilever at point C is:

$$
\begin{equation*}
y_{C}=\frac{P(a+b)^{3}}{3 E I} . \tag{1}
\end{equation*}
$$

[1b] (3 pts) Use the method of MOMENT AREA to show that the vertical deflection of the cantilever at point $B$ is:

$$
\begin{equation*}
y_{B}=\frac{P a^{2}}{6 E I}[3 b+2 a] \tag{2}
\end{equation*}
$$

[1c] (3 pts) Use the method of VIRTUAL FORCES to show that the vertical deflection of the cantilever at point $B$ is:

$$
\begin{equation*}
y_{B}=\frac{P a^{2}}{6 E I}[3 b+2 a] \tag{3}
\end{equation*}
$$

Hint: You can simplify the integration by adopting a transformation $z=a-x$.

Now suppose that a roller support is inserted below point B as follows:


Figure 2: Front elevation view of a cantilever supported by a roller at point B.
[1d] (3 pts) Show that the vertical support reaction at B is:

$$
\begin{equation*}
V_{b}=\frac{P}{2}\left[\frac{3 b+2 a}{a}\right] . \tag{4}
\end{equation*}
$$

[1e] (3 pts) Hence, derive a simple expression for the bending moment at A .

Finally, let's replace the roller support below point B with a spring.


Figure 3: Cantilever supported by a spring at point B.
[1f] (3 pts) Show that the support reaction, $V_{b}$, is now given by the equation:

$$
\begin{equation*}
V_{b}\left[\frac{1}{k}+\frac{a^{3}}{3 E I}\right]=\frac{P a^{2}}{6 E I}[3 b+2 a] \tag{5}
\end{equation*}
$$

[1g] (2 pts) Explain why $V_{b}$ for spring support (i.e., equation 5) is always lower than for roller support (i.e., equation 4).

Question 2: 10 points

OPTIONAL: Moment-Area Method. Consider the cantilevered beam structure shown in Figure 4.


Figure 4: Front elevation view of a cantilevered beam structure.
[2a] (4 pts) Use the method of MOMENT AREA to compute the rotation at point A.
[2b] (4 pts) Use the method of MOMENT AREA to compute the vertical deflection of the beam at point C.
[2c] (2 pts) Draw the deflected shape of the beam. Indicate the sections of beam where the curvature is constant.

Question 3: 10 points

OPTIONAL: Structural Analysis of a Simple Beam Structure. The beam structure shown in Figure 5 supports a uniformly distributed load w (N/m) between points B and C.


Figure 5: Front elevation view of a simple beam structure.
[3a] (4 pts) Use the method of VIRTUAL DISPLACEMENTS to compute formulae for the vertical reactions at A and D. Show all of your working.
[3b] (6 pts) Use the method of VIRTUAL DISPLACEMENTS to compute a formula for the bending moment at C. Show all of your working.

## Question 4: 10 points

OPTIONAL: Principle of Virtual Work. Figure 6 is a front elevation view of a simple truss that supports vertical loads at nodes C and D. All of the truss members have cross section properties AE.


Figure 6: Front elevation view of a simple truss.
[4a] (5 pts). Compute the support reactions and distribution of forces throughout the structure.
[4b] (5 pts). Use the method of VIRTUAL FORCES to show that the total deflection at node C is:

$$
\begin{equation*}
\triangle=\frac{P L}{A E}\left[\frac{8 \sqrt{10}}{3}\right] \tag{6}
\end{equation*}
$$

## Question 5: 10 points

OPTIONAL: Principle of Virtual Work. Consider the supported cantilevered beam structure shown in Figure 7.


Figure 7: Front elevation view of a supported cantilevered beam structure.

Use the principle of VIRTUAL FORCES to compute the two-by-two flexibility matrix connecting displacements at points B and C to applied loads $P_{b}$ and $P_{c}$, i.e.,

$$
\left[\begin{array}{l}
\triangle_{b}  \tag{7}\\
\triangle_{c}
\end{array}\right]=\left[\begin{array}{ll}
f_{11} & f_{12} \\
f_{21} & f_{22}
\end{array}\right]\left[\begin{array}{l}
P_{b} \\
P_{c}
\end{array}\right]
$$

Question 5 continued

## Question 6: 10 points

OPTIONAL: Compute Support Reactions in a Suspension Bridge. Figure 8 is a front elevation view of a suspension bridge that has three spans, two support towers, and anchor supports at the bridge endpoints.


Figure 8: Elevation view of a three-span suspension bridge.

Spans 1, 2 and 3 have lengths $L, 4 L$ and $L$, respectively. Spans 1 and 3 carry a uniform load $w(\mathrm{~N} / \mathrm{m})$. Span 2 carries a uniform load $2 w(\mathrm{~N} / \mathrm{m})$. The towers have height L above the bridge deck level. At the mid-point of Span 2, the lowest point of the cable profile is $k L$ above the bridge deck.

The purpose of this question is to work step by step toward the computation of vertical support reactions at the cable anchor points (i.e., at points A and D ), and at the towers (i.e., at points B and C ). For the purposes of this analysis, assume that the bridge cable weight is bundled into the applied loads $w$, and can otherwise be ignored. Also, assume that the cable passes through the top of the towers on a frictionless support and, as a result, the horizontal component of cable force will be constant along the entire bridge.
[6a] (2 pts) By examining equilibrium of the cable profile in Span 2, show that the horizontal component of cable force is:

$$
\begin{equation*}
H=\left[\frac{4}{1-k}\right] w L \tag{8}
\end{equation*}
$$

[6b] (3 pts) Show that the equation of the cable profile in Span 1 is:

$$
\begin{equation*}
y(x)=\left[\frac{1-k}{8 L}\right] x^{2}+\left[\frac{7+k}{8}\right] x \tag{9}
\end{equation*}
$$

[6c] (3 pts) Use the results from parts [6a] and [6b] to show that the vertical reaction at the anchor support (i.e., at point A ) is:

$$
\begin{equation*}
V_{A}=\left[\frac{7+k}{2-2 k}\right] w L \tag{10}
\end{equation*}
$$

acting downwards.
[6d] (2 pts) Hence, show that the vertical support reaction at Tower 1 is:

$$
\begin{equation*}
V_{B}=\left[\frac{17-9 k}{2-2 k}\right] w L \tag{11}
\end{equation*}
$$

acting upwards.

## BONUS PROBLEM: 4 points

Problem: Figure 9 shows a right triangle ACE that encloses a square BCDF.


Figure 9: Schematic of a right triangle enclosing a square.

Derive a simple formula for the area of BCDF in terms of dimensions $a(\mathrm{AB})$ and $b(\mathrm{EF})$. Show all of your working.

Demonstrate that your formula works for the case $a=1$ and $b=\sqrt{2}$.

