Department of Civil and Environmental Engineering, Fall Semester, 2019

ENCE 353 Final Exam, Open Notes and Open Book

Name: AUSTIN.

Exam Format and Grading. The exam will be 2 hrs plus five minutes to read the questions.

Answer question 1. Then answer **three of the five** remaining questions.

Only the first four questions that you answer will be graded, so please **cross out the two questions you do not want graded** in the table below. Partial credit will be given for partially correct answers, so please show all your working.

After you have finished working on the exam, look at the bonus problem for additional credit. No partial credit for this part of the exam.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
Bonus	4	
Total	50	

Question 1: 20 points

COMPULSORY: Analysis of a Supported Cantilever with Moment Area and Method of Virtual Forces. Consider the cantilevered beam structure shown in Figure 1.

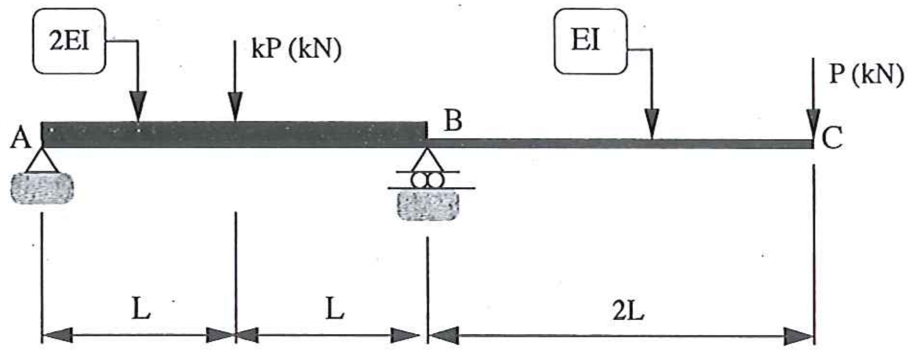
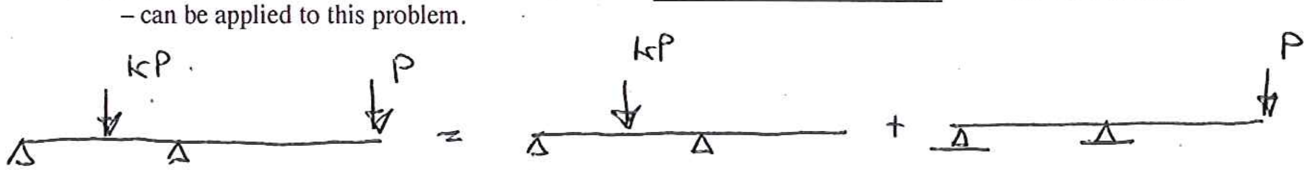


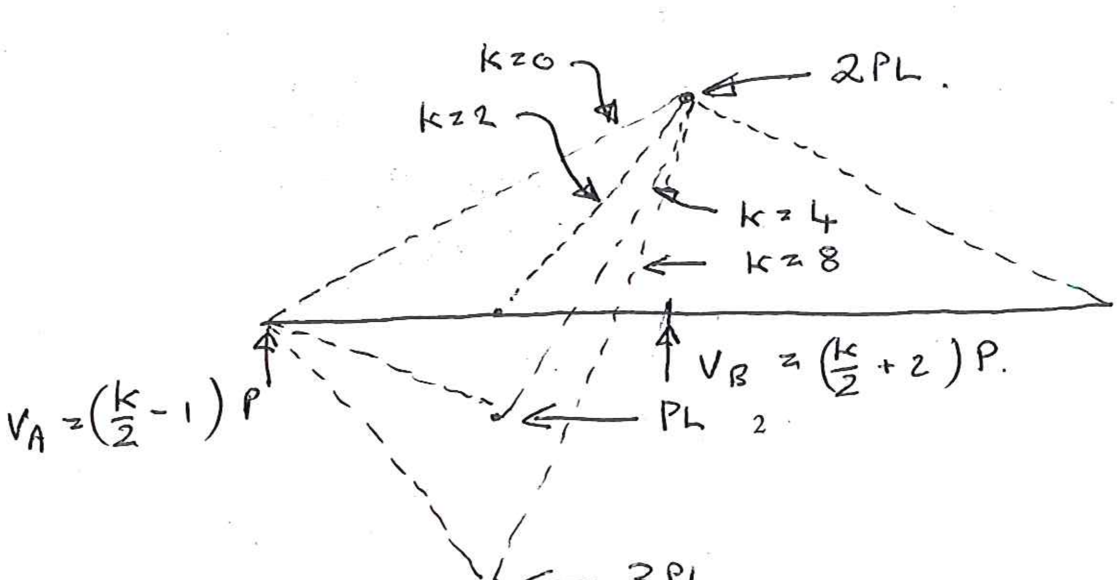
Figure 1: Front elevation view of a cantilevered beam structure.

Segments A-B and B-C have section properties 2EI and EI, respectively. A vertical load P (kN) is applied at point C. A second vertical load of kP (kN) is applied at the mid-span of A-B.

[1a] (2 pts) Draw and label a diagram showing how the <u>**principle of superposition**</u> — hint, hint, hint hint! – can be applied to this problem.



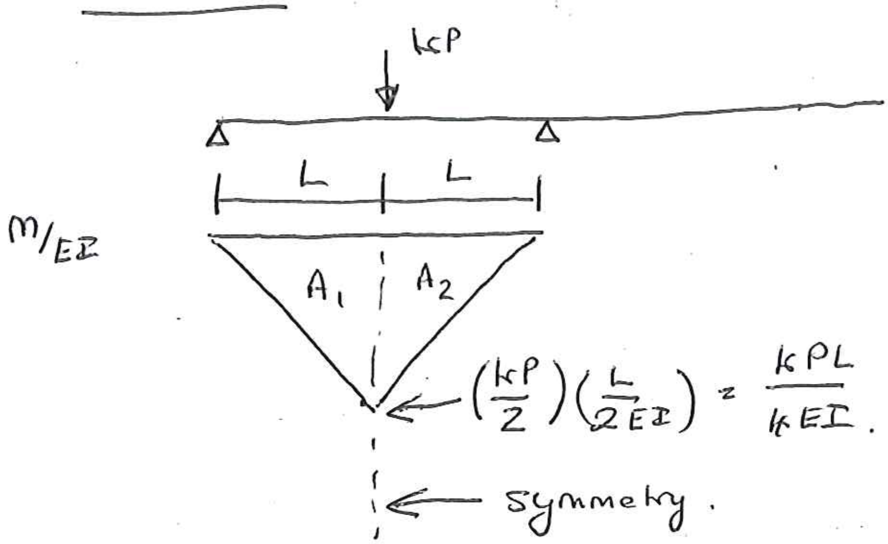
[1b] (2 pts) Draw and label a diagram that shows how the bending moment changes along the beam for various values of k. I suggest that you draw the bending moment diagram for k = 0, and then annotate it to show changes for k = 2, k = 4, and k = 8.



[1c] (4 pts) Use the method of **moment area** to show that the clockwise rotation of point A is:

$$\theta_A = \left[\frac{3k-8}{24}\right]\frac{PL^2}{EI}. \tag{1}$$

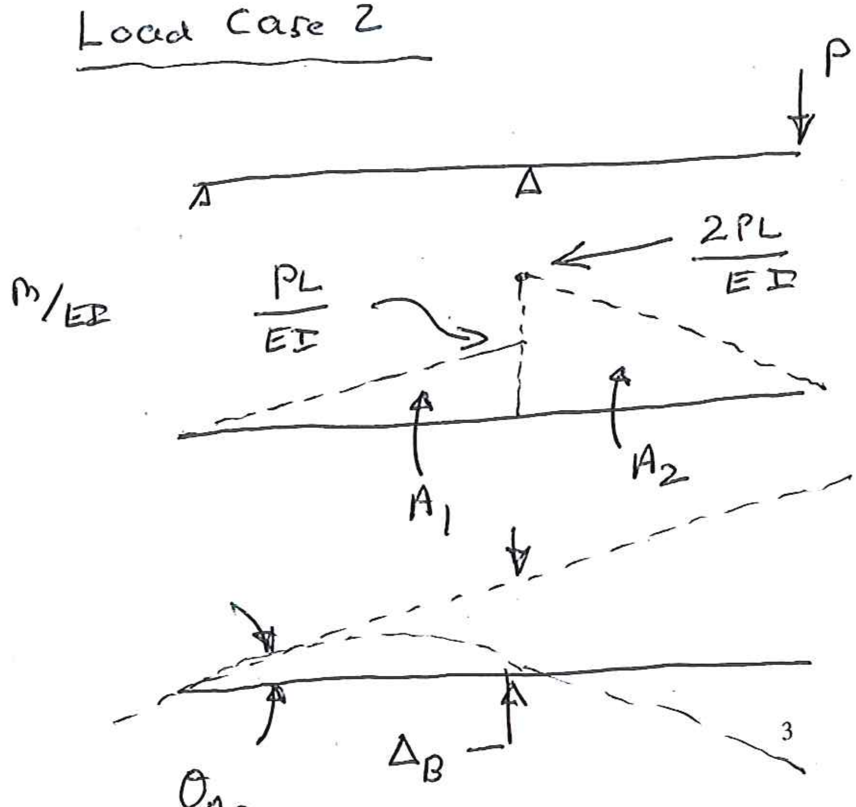
Load Case 1



$$A_1 = A_2 = \frac{1}{2}\left(\frac{kPL}{4EI}\right)L = \frac{kPL^2}{8EI}.$$

$$\theta_{A_{kP}} = \frac{kPL^2}{8EI}.$$

Load Case 2



$$A_1 = \frac{1}{2}\frac{PL}{EI}(2L) = \frac{PL^2}{EI}$$

$$A_2 = 2A_1 = \frac{2PL^2}{EI}.$$

Apply moment area

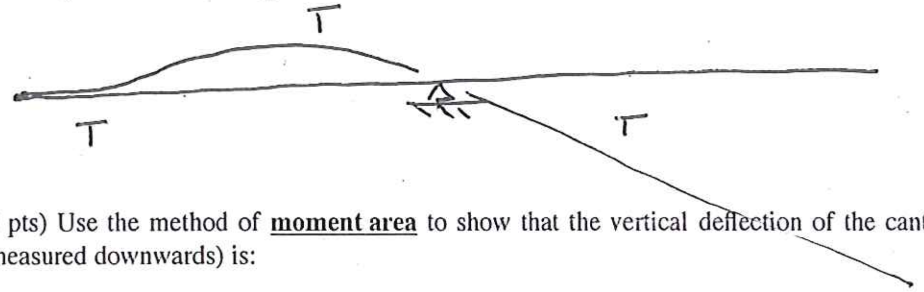
$$\Delta_B = A_1\left(\frac{2}{3}L\right) = \frac{2}{3}\frac{PL^3}{EI}$$

$$\Rightarrow \theta_{A_P} = \left(\frac{\Delta_B}{2L}\right) = \frac{PL^2}{3EI}.$$

Clockwise Rotation $\quad \theta_A = \theta_{A_{kP}} - \theta_{A_P} = \frac{PL^2}{EI}\left[\frac{k}{8} - \frac{1}{3}\right].$

[1d] (3 pts) Draw and label the deflected shape of the beam when k = 8/3. Indicate sections of beam where the fibre is in tension/compression, and where the curvature is zero.

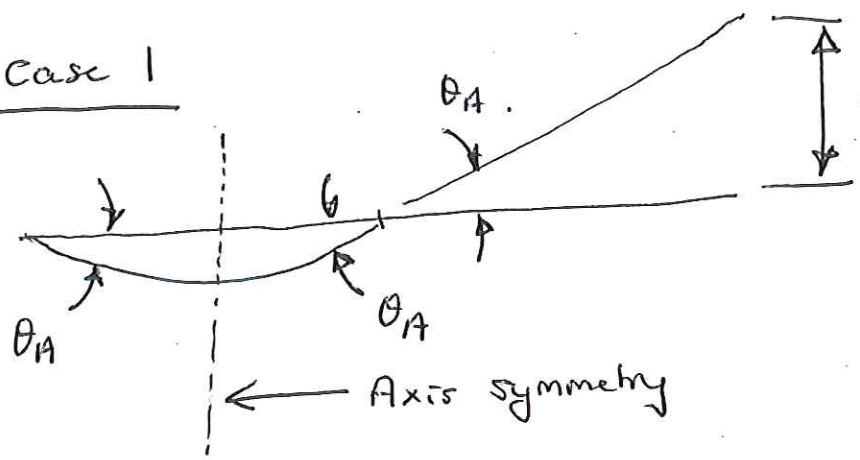
When $k = 8/3$, $V_A = \left(\frac{8}{6} - 1\right)P = \frac{P}{3}$, $\theta_A = 0$



[1e] (5 pts) Use the method of **moment area** to show that the vertical deflection of the cantilever at C (measured downwards) is:

$$\Delta_C = \frac{PL^3}{EI}\left[4 - \frac{k}{4}\right]. \tag{2}$$

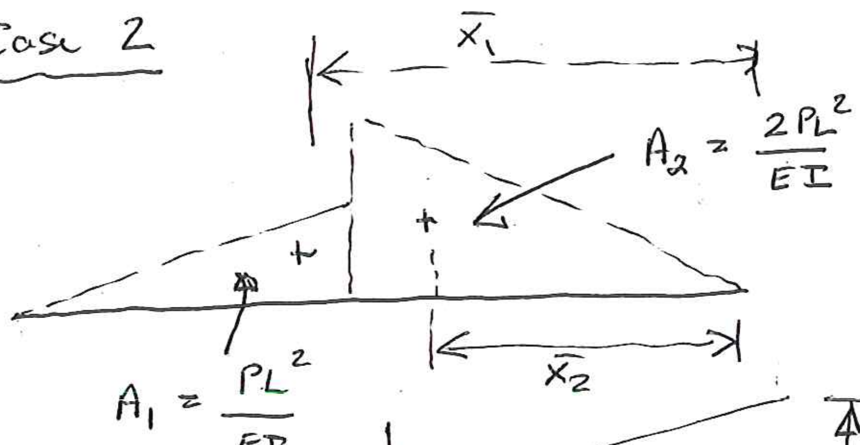
Load Case 1

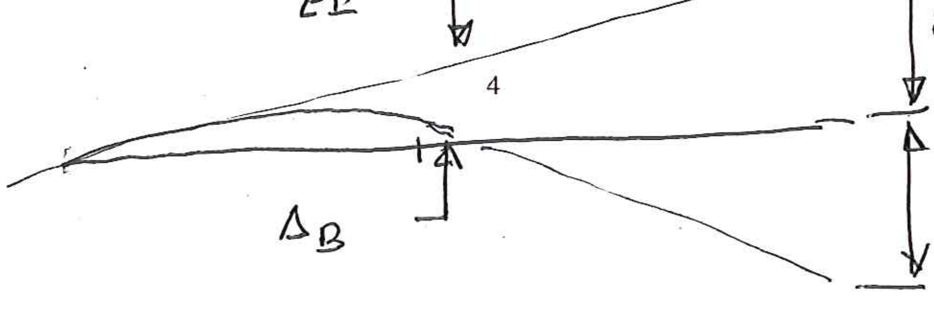


$\Delta_{kP} = (2L)\,\theta_A = \frac{kPL^3}{4EI} \uparrow$.

Load Case 2

M/EI





$2\Delta_B = \frac{4}{3}\frac{PL^3}{EI}$

Question 1e continued:

$$\bar{x}_1 = \left(2 + \frac{2}{3}\right)L = \frac{8}{3}L$$

$$\bar{x}_2 = 4/3 L.$$

Apply moment area

$$\Delta_C + 2\Delta_B = A_1 \bar{x}_1 + A_2 \bar{x}_2$$

$$= \left(\frac{PL^2}{EI}\right)\left(\frac{8}{3}L\right) + \left(\frac{2PL^2}{EI}\right)\left(\frac{4}{3}L\right)$$

$$= \frac{16}{3}\frac{PL^3}{EI}.$$

$$\Rightarrow \Delta_C = \frac{PL^3}{EI}\left[\frac{16}{3} - \frac{4}{3}\right] = \frac{4PL^3}{EI}.$$

Net downwards deflection:

$$\Delta_C = \frac{4PL^3}{EI} - \frac{k}{4}\frac{PL^3}{EI} = \frac{PL^3}{EI}\left[4 - \frac{k}{4}\right].$$

[1f] (4 pts) Now suppose that you wish use the method of **virtual forces** to verify equation 2.

Develop a step-by-step procedure – labeled diagrams and equations would be good – for a solution to this problem. **DO NOT** evaluate the integrals.

Step 1 : Apply unit load at C. Compute BMD.

BMD.

$m(x) = x$

$\leftarrow 2L.$

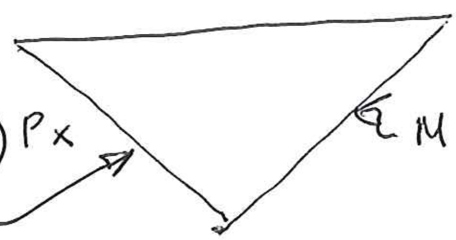
$m(x) = (4L - x)$
$= x_2.$

1

$\rightarrow x$

$x_2 \leftarrow$

Step 2 Find $m(x)$ for load case 1



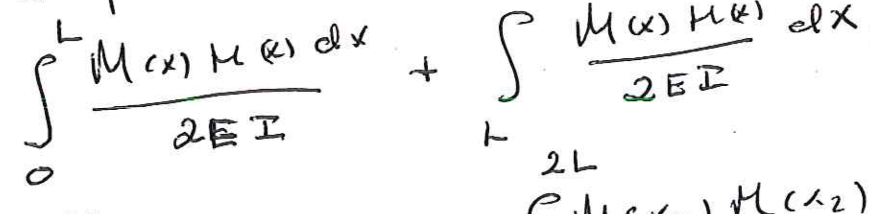
$M(x) = \left(\frac{k}{2} - 1\right) Px$

$M(x) = \left(\frac{k}{2} - 1\right) P(2L - x).$

Step 3 Find $M(x)$ for load case 2

$M(x) = P(4L - x)$
$= P(x_2).$

$M(x) = Px.$



Step 4: Compute & sum deflections.

$$\Delta_1 = \int_0^L \frac{M(x)\, m(x)\, dx}{2EI} + \int_L^{2L} \frac{M(x)\, m(x)}{2EI}\, dx + \int_{2L}^{4L} \frac{M(x) \cdot 0}{EI}$$

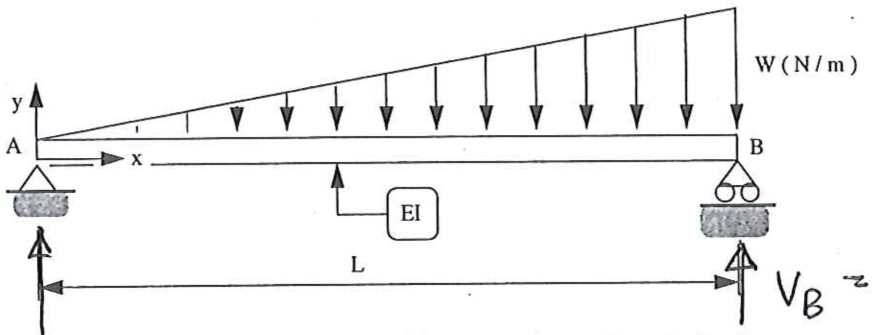
$$\Delta_2 = \int_0^{2L} \frac{M(x)\, m(x)}{2EI}\, dx + \int_0^{2L} \frac{M(x_2)\, m(x_2)}{EI}\, dx.$$

$$\Delta_C = \Delta_1 + \Delta_2.$$

Question 2: 10 points

OPTIONAL: Elastic Curve for Beam Deflections. Figure 2 is a front elevation view of a simply supported beam that carries a triangular load.

$V_A = \frac{W_o L}{6}$



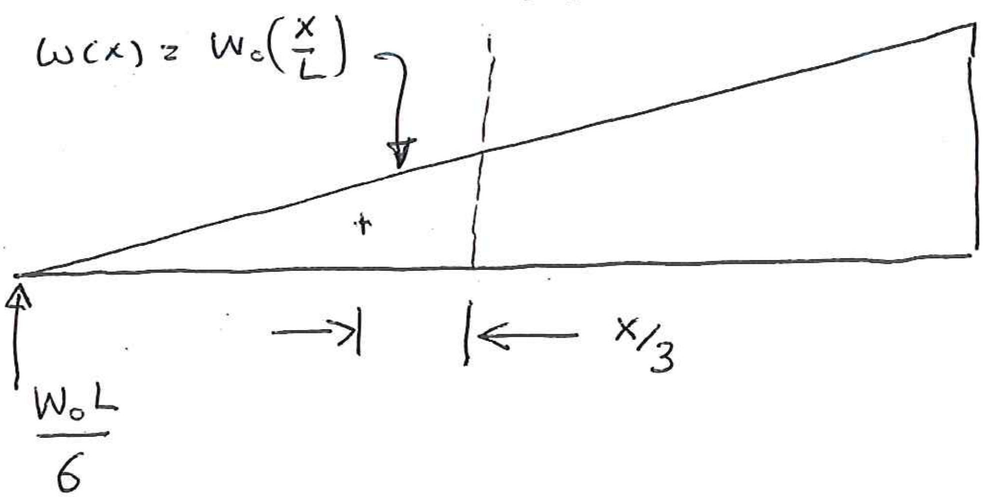
$V_B = \frac{W_o L}{3}$

Figure 2: Simply supported beam carrying a triangular load.

The load increases from zero at point A to W (N/m) at point B. Thus, the total beam loading is WL/2.

[2a] (4 pts). Starting from first principles of engineering, show that the bending moment at point x is:

$$M(x) = \left[\frac{W}{6L}\right] x \left(L^2 - x^2\right). \tag{3}$$



$$M(x) = \left(\frac{W_o L}{6}\right) x - \frac{1}{2}\left(\frac{W_o x^2}{L}\right)\cdot\left(\frac{x}{3}\right)$$

$$= \left(\frac{W_o}{6L}\right) x \left(L^2 - x^2\right).$$

[2b] (4 pts). Show that the elastic curve for beam deflection is given by (notice that in Figure 3, the y axis is pointing upwards):

$$y(x) = \left[\frac{-W}{6LEI}\right]\left[\frac{L^2x^3}{6} - \frac{x^5}{20} - \frac{14L^4x}{120}\right] \tag{4}$$

$$\frac{d^2y}{dx^2} = \frac{-M(x)}{EI} \Rightarrow (EI)\frac{d^2y}{dx^2} = \frac{-W_0}{6L}\left(L^2x - x^3\right)$$

Integrating twice & applying boundary conditions

$y(0) = 0 \rightarrow B = 0$

$y(L) = 0 \rightarrow A = \left(\frac{L^5}{20} - \frac{L^5}{6}\right)\frac{1}{L}$

$= \frac{-14L^4}{120}$.

$$\Rightarrow y(x) = \frac{-W_0}{6EIL}\left[\frac{L^2x^3}{6} - \frac{x^5}{20} - \frac{14L^4x}{120}\right].$$

[2c] (2 pts). Show that the maximum beam curvature occurs at x = $L/\sqrt{3}$.

$\phi = \frac{M(x)}{EI}$, Max $\phi \rightarrow \frac{dM}{dX} = 0$

$\rightarrow L^2 - 3x^2 = 0$

$\boxed{X = L/\sqrt{3}}$

Question 3: 10 points

OPTIONAL: Structural Analysis of a Simple Beam Structure. The beam structure shown in Figure 3 supports an external load P at points B and C.

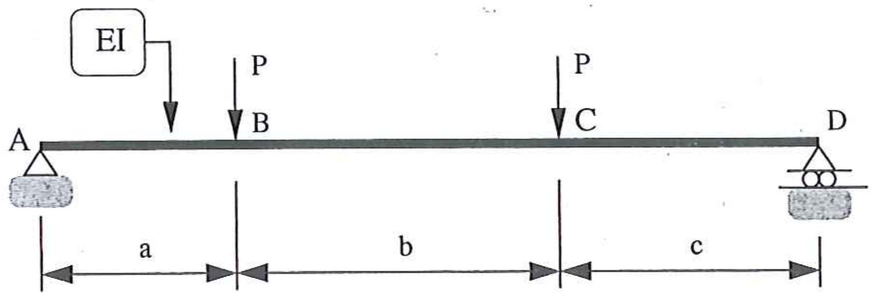
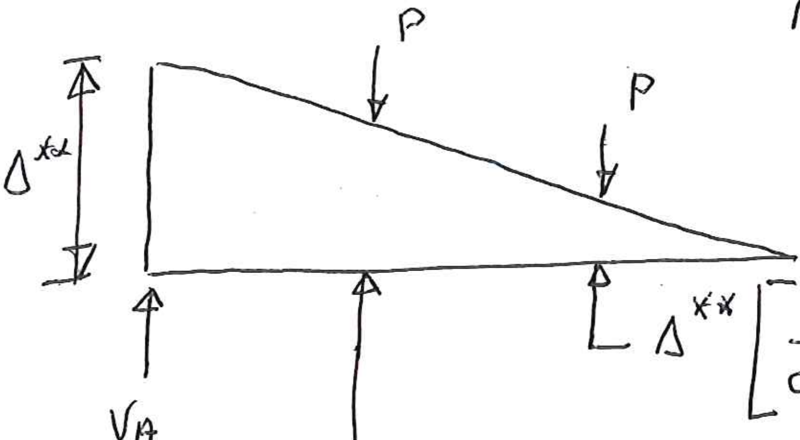


Figure 3: Front elevation view of a simple beam structure.

[3a] (4 pts) Use the method of **virtual displacements** to compute formulae for the vertical reactions at A and D. Show all of your working.

$$V_A \Delta^{**} - \left(\frac{c}{a+b+c}\right)\Delta^{**} - \left(\frac{b+c}{a+b+c}\right)$$

$$\Rightarrow V_A = \left(\frac{b+2c}{a+b+c}\right)P.$$



$$\Delta^{**}\left[\frac{c}{a+b+c}\right]$$

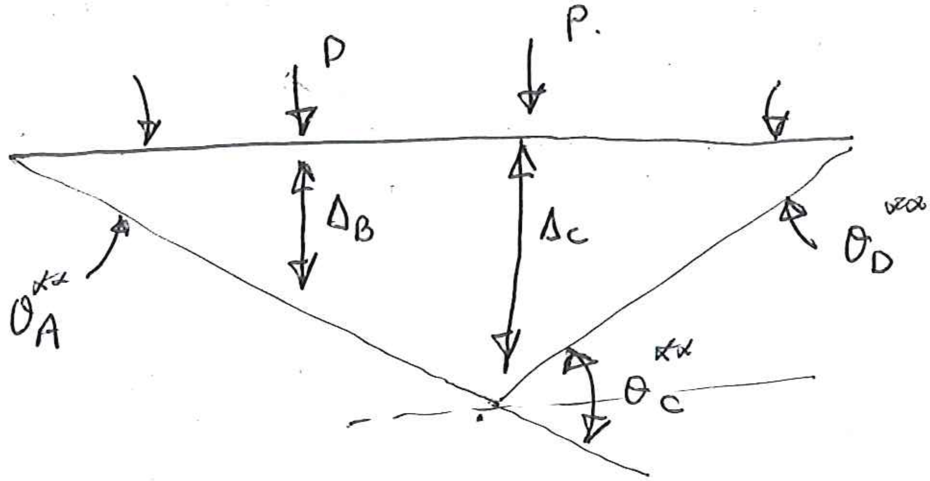
$$\left[\frac{b+c}{a+b+c}\right]\Delta^{**}$$

Similarly, $V_D = \left(\frac{2a+b}{a+b+c}\right)P.$

Check equilibrium

$$V_A + V_D = \left(\frac{2a+2b+2c}{a+b+c}\right)P = 2P.$$

[3b] (6 pts) Use the method of **virtual displacements** to compute a formula for the bending moment at C. Show all of your working.



From geometry:

$$\theta_C^{**} = \theta_A^{**} + \theta_D^{**}.$$

$$\Delta_C = \theta_A^{**}(a+b)$$

$$= \theta_D^{**}\, c$$

$$\Delta_B = \frac{a}{(a+b)}\Delta_C$$

$$\theta_A = \frac{\Delta_C}{(a+b)}$$

$\sum' EWD = 0.$

$$M_C\,\theta_C^{**} = P\Delta_B + P\Delta_C.$$

$$\Rightarrow M_C\left(\theta_a^{**} + \theta_D^{**}\right) = P\left(\frac{a}{a+b} + 1\right)\Delta_C$$

$$\left(\frac{\Delta_C}{a+b}\right) \quad \left(\frac{\Delta_C}{c}\right) = P\left(\frac{2a+b}{a+b}\right)\Delta_C$$

$$\Rightarrow M_C = \left[\frac{(2a+b)c}{(a+b+c)}\right]P.$$

Question 4: 10 points

OPTIONAL: Simple Three-Pinned Arch. Figure 4 is a front elevation view of a simple three-pinned arch that carries a total snow loading of 3WL uniformly distributed over its upper section.

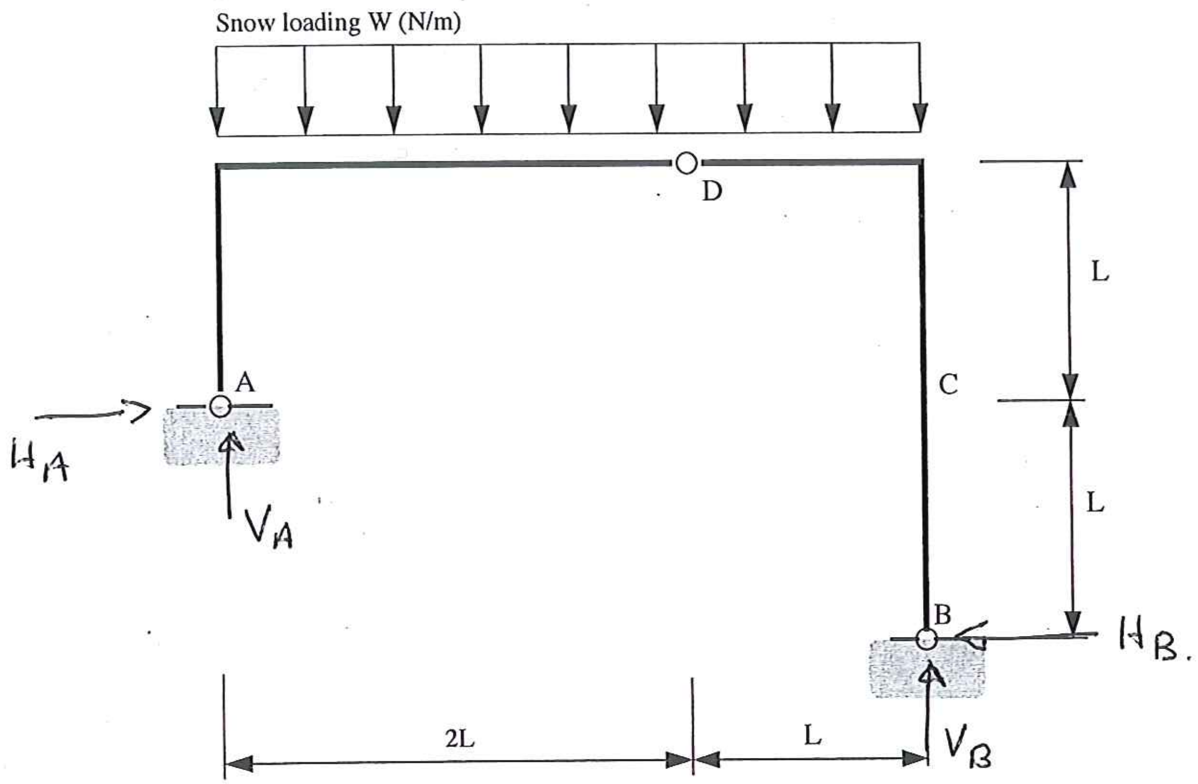


Figure 4: Front elevation view of a three-pinned arch that supports a snow loading.

[4a] (6 pts) Compute the vertical and horizontal components of reaction force at supports A and B as a function of W and L.

$\sum V = 0 \quad \rightarrow \quad V_A + V_B = 3WL. \quad$ — (A)

$\sum H = 0 \quad \rightarrow \quad H_A = H_B \quad$ — (B).

$\sum M_D = 0 \text{ (LHS)} \quad (2WL)L + H_A L = 2L V_A$

$\Rightarrow 2WL + H_A = 2V_A \quad$ — (c)

$\sum M_D = 0 \text{ (RHS)} \quad (WL)(\frac{L}{2}) + H_B(2L) = V_B L$

$\rightarrow WL + 4H_B = 2V_B \quad$ — (D)

Queation 4a continued:

Add (C) + (D) ; ~~into~~ insert (B)

$2wL + H_A + wL + 4H_B = 2(V_A + V_B) = 6wL$

$\Rightarrow 3wL + 5H_A = 6wL$

$\Rightarrow H_A = H_B = 3/5\, wL.$ —— (E)

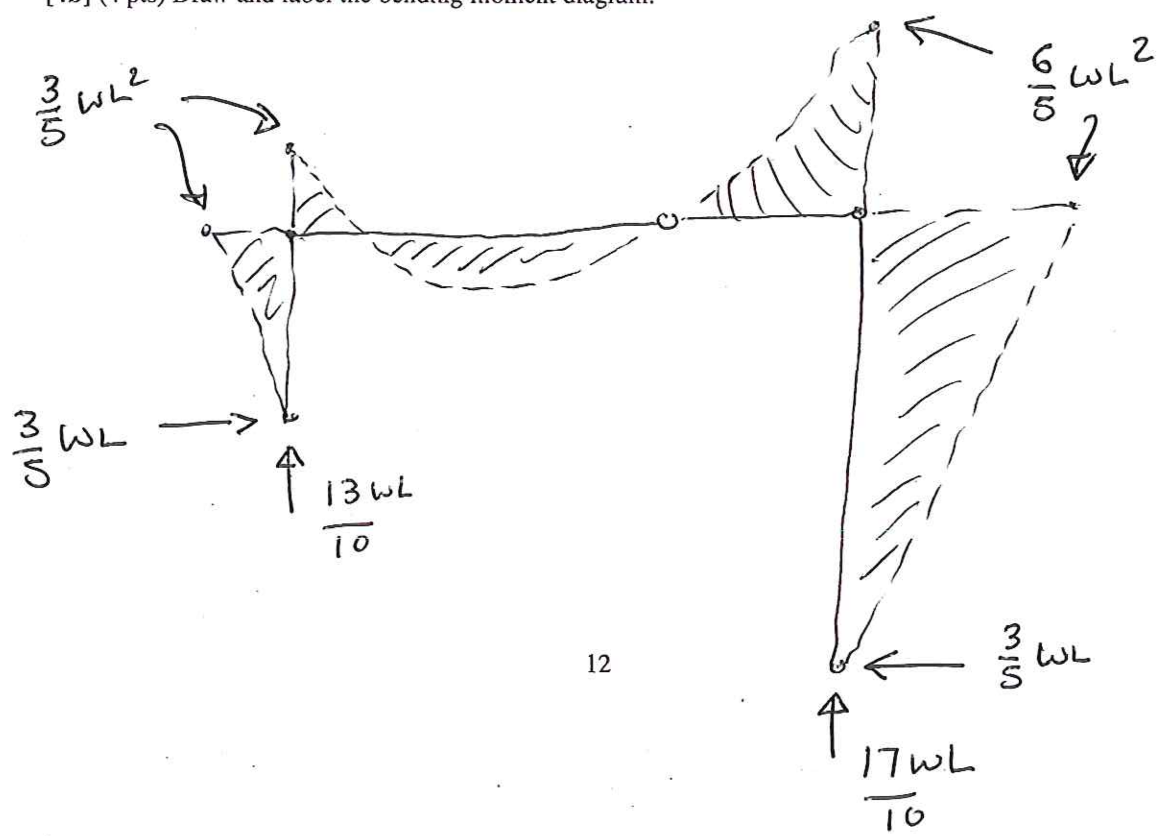
Plug (E) into (C) & (D)

$\Rightarrow V_A = \frac{13}{10} wL, \quad V_B = \frac{17}{10} wL.$

Check equlibrium.

$V_A + V_B = \frac{13\, wL}{10} + \frac{17}{10} wL = 3wL$ ✓.

[4b] (4 pts) Draw and label the bending moment diagram.



Question 5: 10 points

OPTIONAL: Principle of Virtual Work. Figure 5 is a front elevation view of a simple truss that supports vertical loads at nodes C and D. All of the truss members have cross section properties AE.

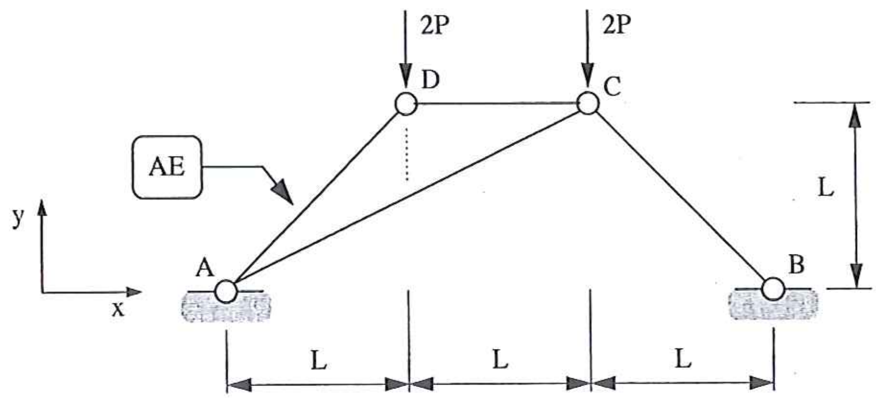
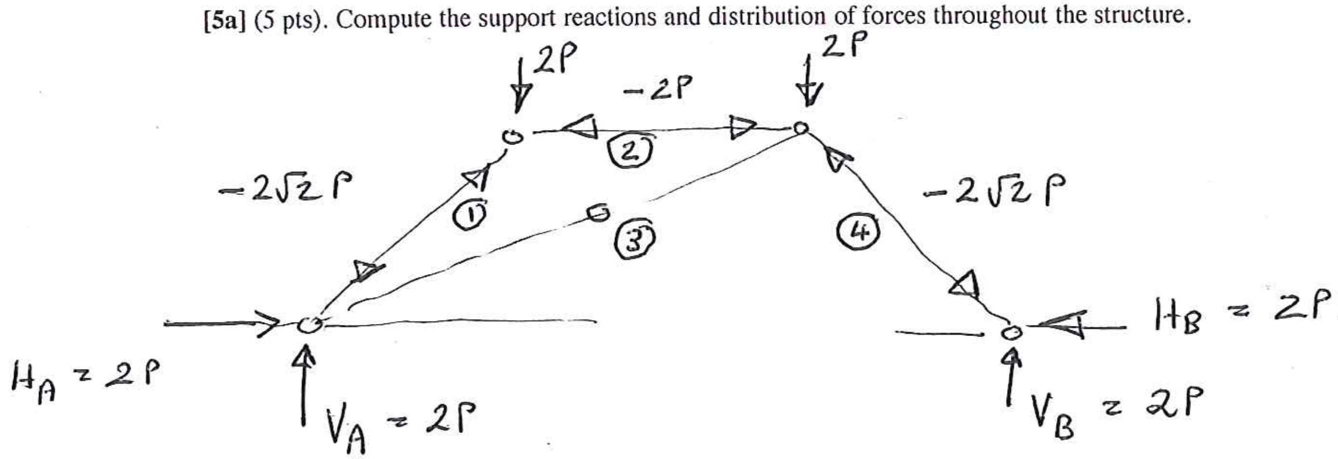
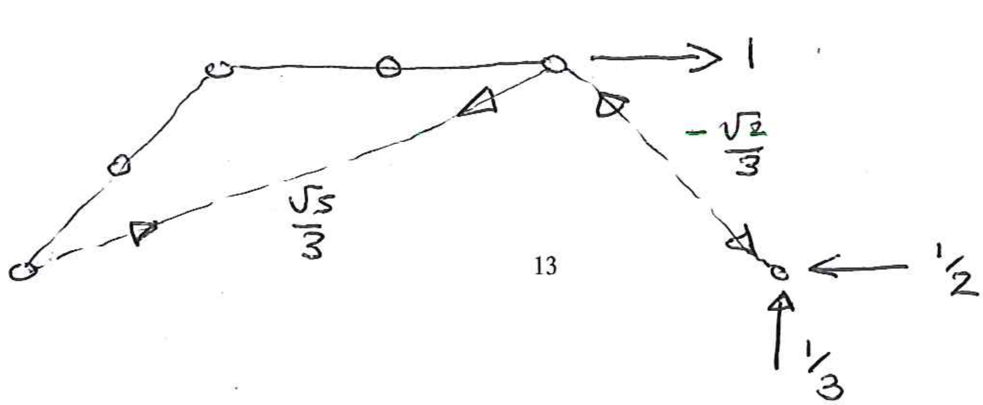


Figure 5: Front elevation view of a simple truss.

[5a] (5 pts). Compute the support reactions and distribution of forces throughout the structure.



Apply unit load at C



[5b] (5 pts). Use the method of **virtual forces** to show that the horizontal deflection at node C is:

$$\Delta = \frac{PL}{AE}\left[\frac{4\sqrt{2}}{3}\right]. \tag{5}$$

Assemble table.

Element	L/AE	F	f	$\frac{FfL}{AE}$
1	$\sqrt{2}L/AE$	$-2\sqrt{2}P$	0	0
2	L/AE	$-2P$	0	0
3	$\sqrt{5}L/AE$	0	$\sqrt{5}/3$	0
4	$\sqrt{2}L/AE$	$-2\sqrt{2}P$	$\frac{-\sqrt{2}}{3}$	$\frac{4\sqrt{2}PL}{3AE}$
				$\frac{4\sqrt{2}PL}{3AE}$

$$\Rightarrow \Delta_H = \frac{PL}{AE}\left[\frac{4\sqrt{2}}{3}\right].$$

Question 6: 10 points

Consider the truss structure shown in Figure 6.

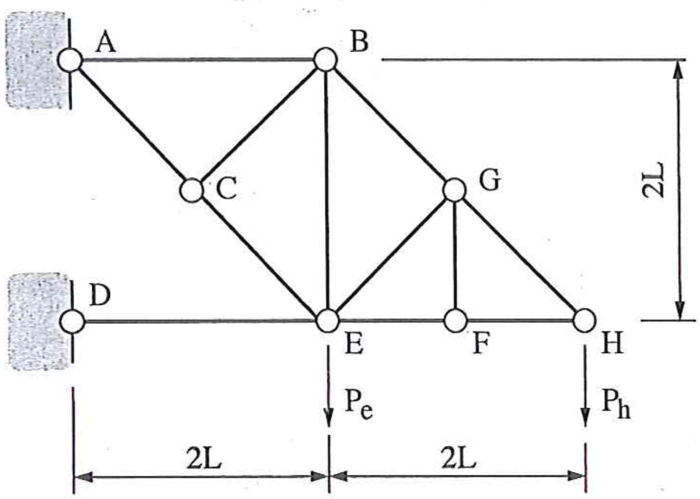


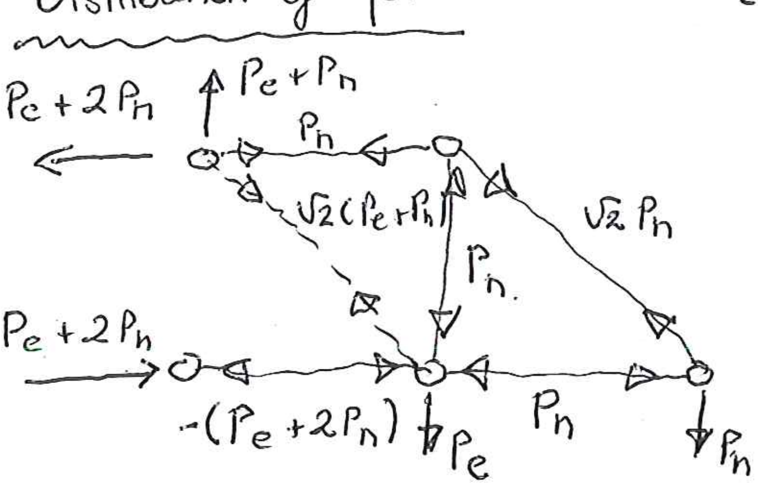
Figure 6: Elevation view of a pin-jointed truss.

The horizontal and vertical degrees of freedom are fully-fixed at supports A and D. The truss carries vertical loads P_e and P_h at nodes E and H, respectively. All frame members have cross section properties AE.

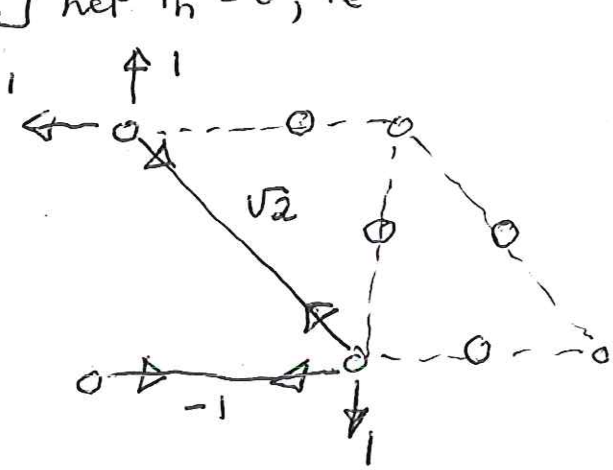
[6a] (2 pts) Use the **method of joints** to identify all of the zero-force members. Label these members on Figure 6.

[6b] (2 pts) Use the method of **virtual forces** to compute the vertical deflection at node E due to load P_e alone (i.e., $P_h = 0$).

Distribution of forces

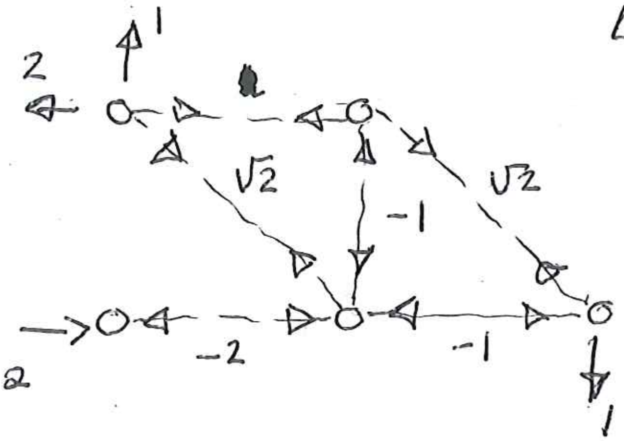


[6b] Let $P_h = 0$, $P_e = 1$



$$\Delta_e = \frac{(\sqrt{2}\,P_e)(\sqrt{2}\,2L)(\sqrt{2})}{AE} + \frac{(-1\,P_e)(-1)\,2L}{AE} = \frac{P_e L}{AE}\left[4\sqrt{2} + 2\right]$$

[6c] (3 pts) Use the method of **virtual forces** to compute the vertical deflection at node H due to load P_h alone (i.e., $P_e = 0$).



$$\Delta_h = \sum_1^1 \frac{F_i f_j L}{AE}$$

$$= \frac{(2P_h L) + \sqrt{2}\sqrt{2}P_h(2\sqrt{2}L) + -2P_h(-2)2L + (-1)(-P_h)2L + \sqrt{2}P_h\sqrt{2}\,2\sqrt{2}L) + -1 - P_h\,2L}{AE}$$

$$= \frac{PL}{AE}\left[2 + 4\sqrt{2} + 8 + 2 + 4\sqrt{2} + 2\right]$$

$$= PL/AE\left[14 + 8\sqrt{2}\right].$$

[6d] (3 pts) Use the principle of virtual forces to compute the two-by-two flexibility matrix connecting the vertical displacements at points E and H to applied loads P_e and P_h, i.e., as a function of P_e, P_h, L and AE.

$$\begin{bmatrix} \Delta_e \\ \Delta_h \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_e \\ P_h \end{bmatrix}. \qquad (6)$$

$$f_{11} = \sum^1 \frac{n_i n_i L}{AE} = \left[0 + 2.2\sqrt{2}L + 0 + 0 + 2L + 0\right]/AE$$

$$= \frac{(2 + 4\sqrt{2})L}{AE}.$$

$$f_{22} = \sum^1 \frac{n_2 n_2 L}{AE} = \frac{L}{AE}\left[14 + 8\sqrt{2}\right].$$

$$f_{12} = \sum^1 \frac{n_i n_2 L}{AE} = \left[0 + 2.2L\sqrt{2} + 0 + 0 + 2.2L + 0\right]/AE$$

$$= \frac{(\sqrt{2}+1)4L}{AE}$$

$$= f_{21}.$$

Question 6 continued ...

$$\Rightarrow \begin{Bmatrix} \Delta_e \\ \Delta_h \end{Bmatrix} = \frac{L}{AE} \begin{bmatrix} (2+4\sqrt{2}) & 4(1+\sqrt{2}) \\ 4(1+\sqrt{2}) & (14+8\sqrt{2}) \end{bmatrix} \begin{Bmatrix} P_e \\ P_h \end{Bmatrix}$$

BONUS PROBLEM: 4 points

Problem: Let a and r be positive numbers with $r < a$. The following equation

$$(x-a)^2 + y^2 = r^2 \tag{7}$$

describes a circle of radius r centered at coordinate $(a, 0)$, i.e.,

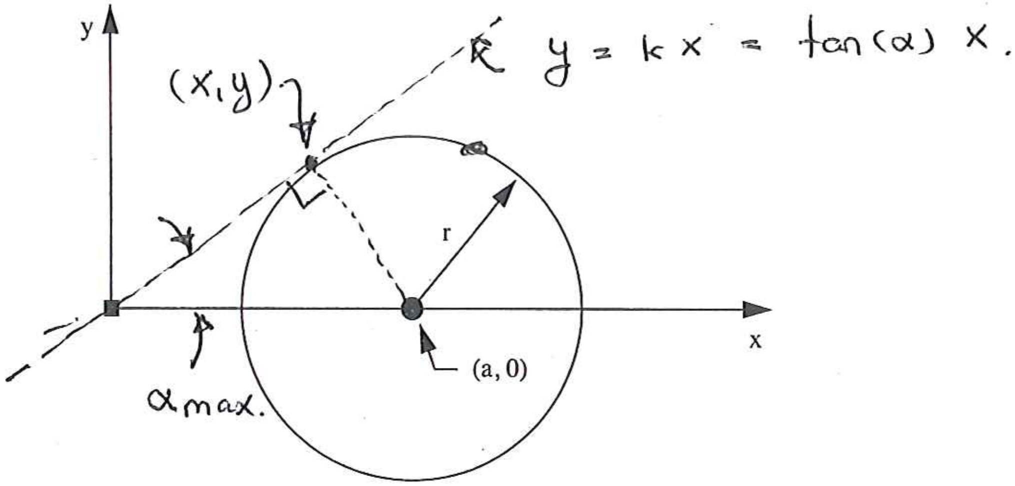


Figure 7: Schematic of circle of radius r centered at $(a, 0)$.

By using simple geometric arguments and nothing more complicated than grade 7-8 math (i.e., no calculus!) prove that the maximum value of y divided by x is:

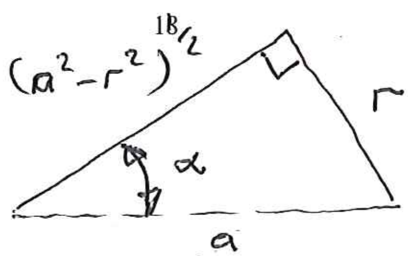
$$\left[\frac{y}{x}\right]_{\text{maximum value}} = \left[\frac{r}{\sqrt{a^2 - r^2}}\right]. \tag{8}$$

Justify your solution with the appropriate equations and rationale.

Let $y = kx$ (straight line in Fig 7). $\left(\frac{y}{x}\right)_{max} = k_{max} = \tan(\alpha)$ ← max!

From geometry, $\tan(\alpha) = \left(\frac{y}{x}\right)$

At max value of k, line is tangent to circle



$\Rightarrow \left(\frac{y}{x}\right)_{max} = \frac{r}{\sqrt{a^2-r^2}}$