Department of Civil and Environmental Engineering, Fall Semester, 2018

ENCE 353 Midterm 2, Open Notes and Open Book

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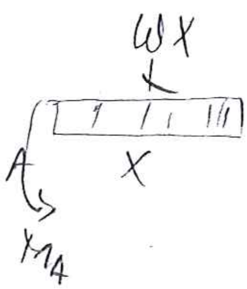
Exam Format and Grading. Attempt all three questions. Partial credit will be given for partially correct answers, so please show all of your working.

Question	Points	Score
1	15	
2	15	
3	10	
Total	40	

Question 1: 15 points

Deriving Formulae for the Deflection of a Cantilever Beam. The cantilever beam structure shown in Figure 1 carries a uniform load w (N/m) along its entire length.



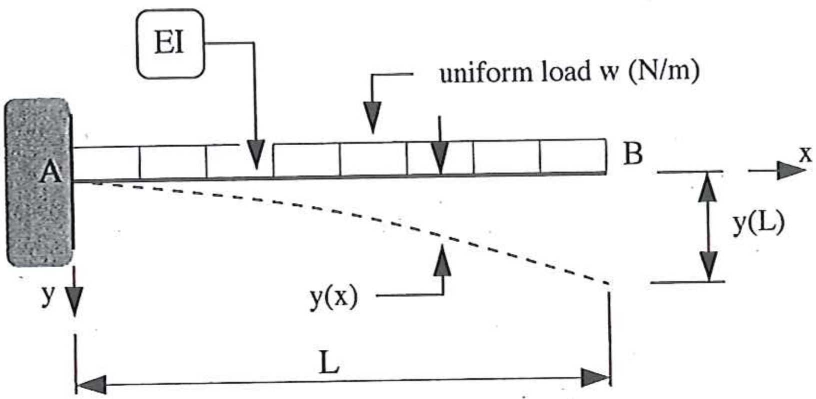


Figure 1: Front elevation view of a cantilever beam carrying a uniform load.

The beam is fully fixed at point A and the flexural stiffness EI is constant along the beam. The coordinate system is positioned at point A.

[1a] (6 pts) Starting from the differential equation,

$$\frac{d^2y}{dx^2} = \left[\frac{M(x)}{EI}\right], \tag{1}$$

and appropriate boundary conditions, show that:

$$y(x) = \left(\frac{w}{24EI}\right)\left(6L^2x^2 - 4Lx^3 + x^4\right). \tag{2}$$

$$M(x) = \frac{W_0}{2}(L-x)^2 = \frac{W_0}{2}\left[L^2 - 2Lx + x^2\right]$$

$$\Rightarrow \quad EI\frac{d^2y}{dx^2} = \frac{W_0}{2}\left[L^2 - 2Lx + x^2\right]$$

$$\Rightarrow \quad EI\frac{dy}{dx} = \frac{W_0}{2}\left[L^2x - Lx^2 + \frac{x^3}{3}\right] + A$$

$$\Rightarrow \quad EI\,y(x) = \frac{W_0}{2}\left[\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right] + Ax + B$$

Question 1a continued:

Boundary conditions.

$dy/dx = 0$ at $x = 0 \Rightarrow A = 0$

$y = 0$ at $x = 0 \Rightarrow B = 0$

$$\Rightarrow y(x) = \frac{W_o}{2EI}\left[\frac{L^2x^2}{2} - \frac{Lx^3}{3} + \frac{x^4}{12}\right]$$

$$= \frac{W_o}{24EI}\left[6L^2x^2 - 4Lx^3 + x^4\right]$$

[1b] (3 pts) Derive a formula for the slope of the beam as a function of x. This is a one line solution. Use your formula to verify that the beam rotation at B is:

$$\theta_B = \left(\frac{wL^3}{6EI}\right). \tag{3}$$

$$\theta_x = \frac{dy}{dx} = \frac{W_o}{6EI}\left[3L^2x - 3Lx^2 + x^3\right] \leftarrow \text{from pt (1a)}$$

$$\Rightarrow \theta_B = \frac{wL^3}{6EI} = \theta_{x=L}.$$

[1c] (6 pts) Using the results of question [1a] as a starting point, compute the support reactions at A and B for the propped cantilever shown in Figure 2

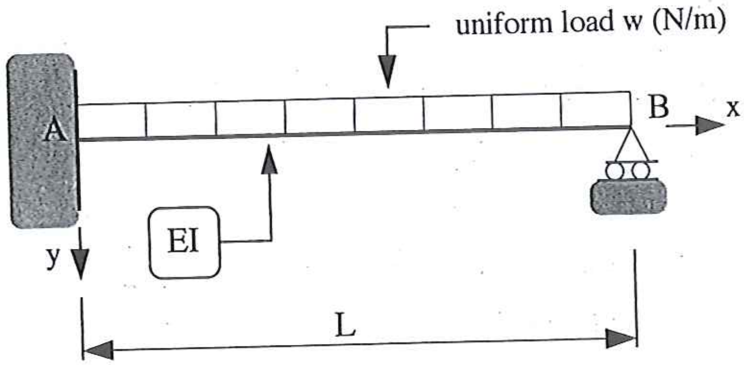


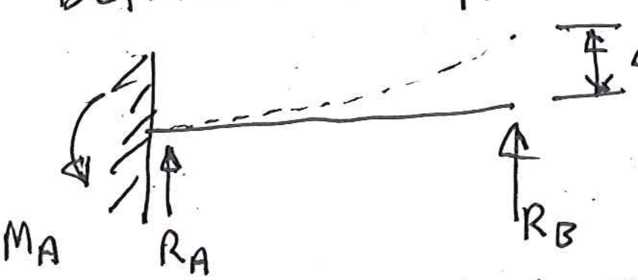
Figure 2: Propped cantilever beam carrying a uniform load.

$$y(x) = \frac{w}{24EI}\left[6L^2x^2 - 4Lx^3 + x^4\right]$$

$$\Rightarrow \quad y(L) = \frac{w_0}{24EI}\left[6L^4 - 4L^4 + L^4\right] = \cancel{\frac{w_0L^4}{24EI}}.$$

$$= \frac{w_0L^4}{8EI}$$

Deflection at support:



$$\Delta_{RB} = \frac{R_B L^3}{3EI}$$

Compatibility of displacements: $\Delta_{RB} = y(L)$

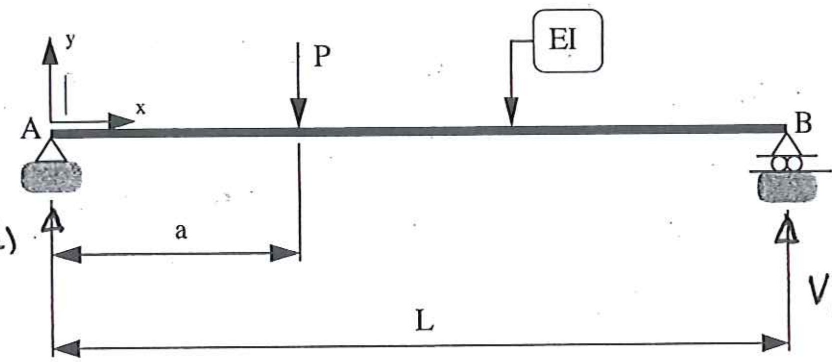
$$\Rightarrow \quad R_B = \frac{3w_0L}{8}.$$

$$R_A + R_B = w_0L \Rightarrow R_A = \frac{5w_0L}{8}.$$

$$M_A = \frac{w_0L^2}{2} - \frac{3w_0L^2}{8} = \frac{w_0L^2}{8}.$$

Question 2: 15 points

Moment-Area and Rotational Deflections. The simple beam shown in Figure 3 has length L and uniform section properties EI. A point load P is applied at distance a from the left-hand support.



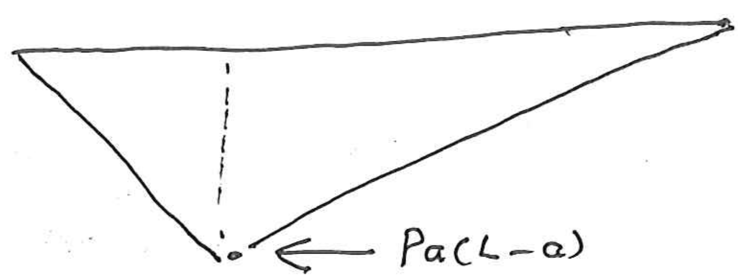
$V_A = \frac{P(L-a)}{L}$

$V_B = \frac{Pa}{L}$

Figure 3: Front elevation view of a simple beam structure.

[2a] (3 pts) Draw and label the M(x)/EI diagram in terms of the problem parameters (i.e., P, EI, L and a).

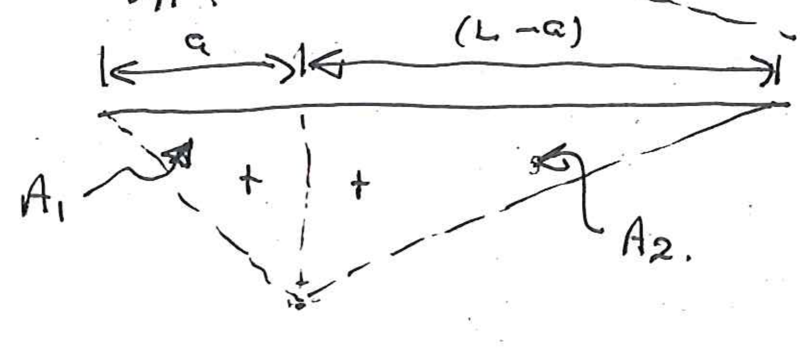
$\underline{M/EI}$



$\frac{Pa(L-a)}{L}$

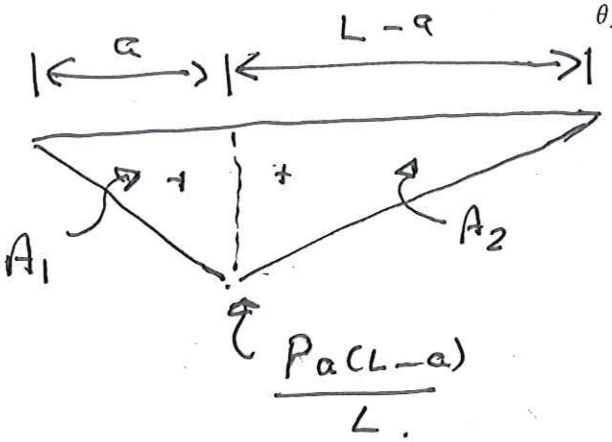
$\underline{\text{Deflection}}$

$\theta_A = \frac{\Delta_B}{L}$



[2b] (8 pts) Use the **method of moment-area** to show that the beam rotation at A is:

$$\theta_A = \left[\frac{P}{6EI}\right]\left[\frac{a(L-a)(2L-a)}{L}\right]. \tag{4}$$



$A_1 = \frac{1}{2}\frac{Pa^2(L-a)}{L}$ $\quad \bar{x}_1 = L - \frac{2}{3}a$

$A_2 = \frac{1}{2}\frac{Pa(L-a)^2}{L}$ $\quad \bar{x}_2 = \frac{2}{3}(L-a)$

Apply moment area.

$$EI\,\Delta_B = A_1\bar{x}_1 + A_2\bar{x}_2$$

$$= \frac{1}{2}\frac{Pa^2(L-a)}{L}\left(L-\frac{2}{3}a\right) + \frac{Pa(L-a)^2}{2L}\,\frac{2}{3}(L-a)$$

$$= \frac{Pa(L-a)}{2L}\left[a\left(L-\frac{2}{3}a\right) + \frac{2}{3}(L-a)^2\right]$$

multiply by $\frac{3}{3}$

$$= \frac{Pa(L-a)}{6L}\underbrace{\left[(3L-2a)a + 2(L-a)^2\right]}_{3La - 2a^2 + 2L^2 - 4La + 2a^2}$$

$\Rightarrow 2L^2 - La = L(2L-a).$

$$\Rightarrow \Delta_B = \frac{Pa(L-a)(2L-a)}{6EI} = \theta_A \cdot L$$

Note: If $a = L/2$

$\theta_A = \frac{PL^2}{16EI}$ ✓.

It works!

$$\Rightarrow \theta_A = \frac{Pa(L-a)(2L-a)}{6EI\,L}$$ ✓

[2c] (4 pts) Show that the **maximum value** of beam rotation at A occurs when:

$$a = L\left[1 - \frac{\sqrt{12}}{6}\right]. \tag{5}$$

From (2b).
$$\left\{\frac{6EI.L}{P}\right\}.\ \theta_A = a(L-a)(2L-a).$$
$$= a(2L^2 - 3La + a^2)$$

At the maximum value. $\frac{\partial \theta_A}{\partial a} = 0 \Rightarrow 2L^2 - 6La + 3a^2 = 0$

$$3a^2 - 6La + 2L^2 = 0$$

$$\Rightarrow \quad a = \frac{6L \pm \sqrt{36L^2 - 4.3.2L^2}}{2\cdot 3}$$

$$= \frac{6L \pm \sqrt{36L^2 - 24L^2}}{6}$$

$$= \frac{6L \pm \sqrt{12}L}{6}$$

Choose root $0 \le a \le L$.

$$\Rightarrow \quad a = \frac{6L - \sqrt{12}L}{6} = \left[1 - \frac{\sqrt{12}}{6}\right]L. \approx 0.422L.$$

help check:

$$6EIL\,\theta_A = P(L-a)(2L-a)a \leftarrow a = 0.422L$$
$$= 0.384\,PL^3$$

Compare: $\theta_{L/} = \frac{1}{16}\frac{PL^2}{EI}$

$$\Rightarrow (\theta_A) \ldots \frac{0.384\,PL^3}{\ldots} \quad 0.0641\,PL^2 \ldots \quad \ldots 0625\,\frac{PL^2}{\ldots EI}$$

Question 3: 10 points

Simple Three-Pinned Arch. Figure 4 is a front elevation view of a simple three-pinned arch. A vertical load P is applied at node D.

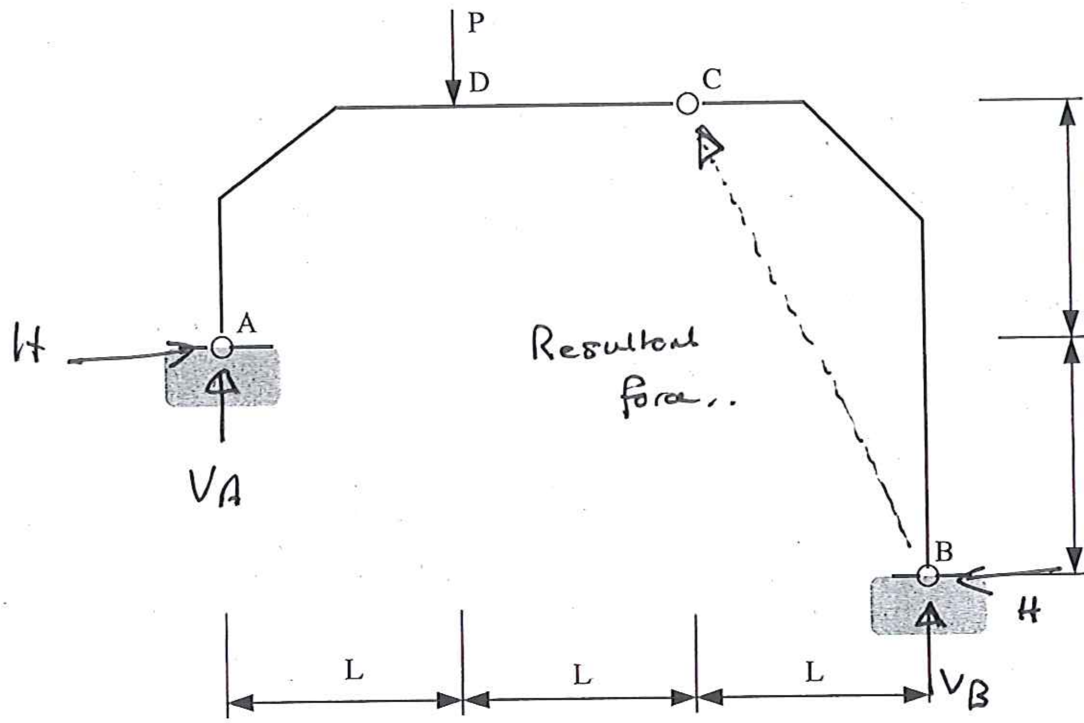


Figure 4: Front elevation view of a simple three-pinned arch.

[3a] (5 pts) Compute the vertical and horizontal components of reaction force at supports A and B as a function of P.

$\sum V = 0 \Rightarrow V_A + V_B = P.$ — (A)

$\sum M_C = 0 \Rightarrow V_B L - H\,2L = 0 \Rightarrow V_B = 2H$ — (B)

$\sum M_C = 0 \Rightarrow PL + HL = V_A\,2L \Rightarrow 2V_A = P + H.$ — (C)

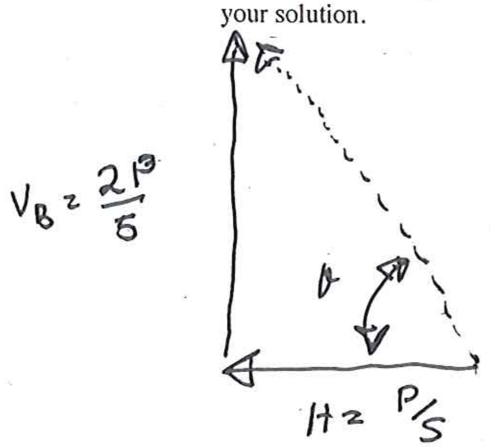
Combining (A) & (B). $\Rightarrow V_A = P - 2H.$ — (D)

From (C) & (D) $2(P - 2H) = P + H \Rightarrow H = P/5.$

$V_B = \frac{2}{5} P$

$V_A = \frac{3}{5} P.$

[3b] (5 pts) Compute the magnitude and orientation of the **total reaction force vector** at support B. Show that it passes through the hinge at C. You can annotate Figure 4 if you think it will help to explain your solution.



$$\tan(\theta) = \frac{V_B}{H} = 2$$