Department of Civil and Environmental Engineering, Fall Semester, 2018

ENCE 353 Final Exam, Open Notes and Open Book

Name: AUSTIN.

Exam Format and Grading. The exam will be 2 hrs plus five minutes to read the questions.

Answer question 1. Then answer **three of the five** remaining questions.

Only the first four questions that you answer will be graded, so please **cross out the two questions you do not want graded** in the table below. Partial credit will be given for partially correct answers, so please show all your working.

After you have finished working on the exam, look at the bonus problem for additional credit. No partial credit for this part of the exam.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
Bonus	5	
Total	50	

Question 1: 20 points

COMPULSORY: Computing Displacements with Moment Area and the Method of Virtual Forces. Figure 1 is a front elevation view of a cantilever beam carrying external loads kP at B and P at C. The parameter k varies over the range 0 through 4.

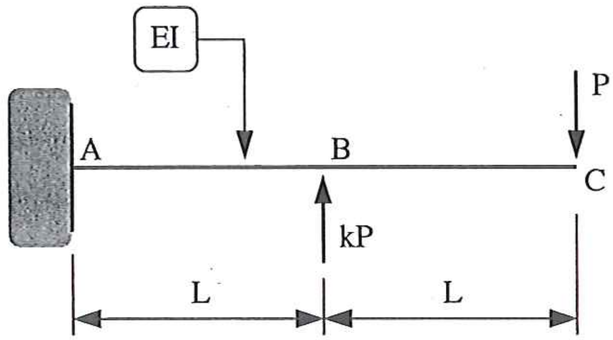
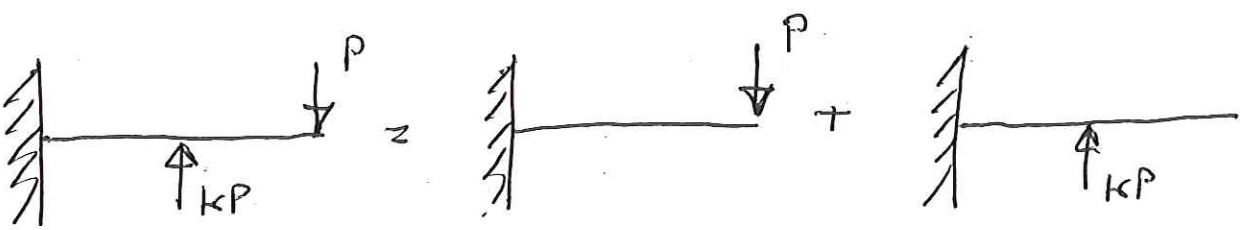


Figure 1: Cantilever beam carrying two applied loads, kP and P.

EI is constant along the cantilever beam. The material properties are linear' and elastic, and displacements are small.

[1a] (2 pts) Briefly explain how the principle of superposition can be applied to this problem.

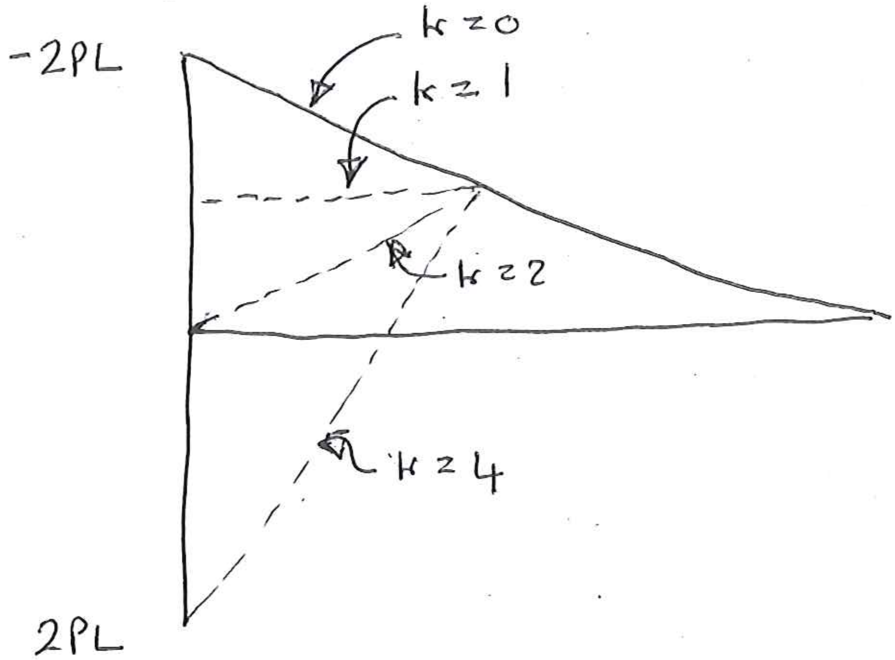


[1b] (2 pts) From a structural analysis standpoint, what does the "small displacements" assumption mean?

(1) Equations of equilibrium are solved with respect to the undeformed configuration

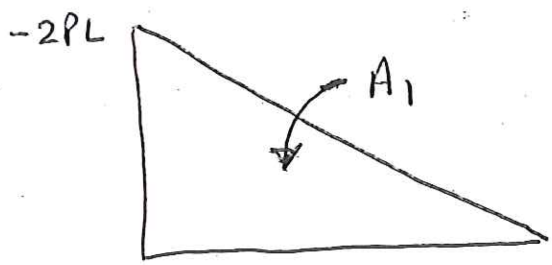
(2) Analysis procedures are simplified: e.g. slopes are small, $(dy/dx)^2 \rightarrow 0$. and $\tan(\theta) \approx \theta + O(\theta^2)$.

[1c] (3 pts) Draw and label a diagram that shows how the bending moment changes along the beam for various values of k. I suggest that you draw the bending moment diagram for $k = 0$, and then annotate it to show changes for $k = 1$, $k = 2$, and $k = 4$.



[1d] (3 pts) Use the method of **MOMENT AREA** to show that the clockwise rotation at point C is:

$$\theta_C = \frac{PL^2}{2EI}\left[4 - k\right] \tag{1}$$

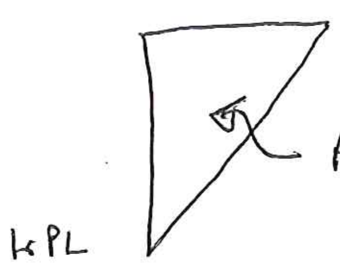


$$A_1 = \frac{1}{2}\,\frac{2PL \cdot 2L}{EI} = \frac{2PL^2}{EI}$$

$$\theta_C = A_1 + A_2$$

+

$$= \frac{4PL^2}{2EI} - \frac{kPL^2}{2EI}$$



$$A_2 = \frac{-1}{2}\,\frac{kPL^2}{EI}$$

$$= \frac{PL^2}{2EI}(4 - k).$$

[1e] (5 pts) Use the method of **MOMENT AREA** to show that the vertical deflection (measured downwards) of the cantilever at C is:

$$\Delta_C = \frac{PL^3}{3EI}\left[8 - \frac{5k}{2}\right]. \qquad (2)$$

$$\Delta_c = A_1 \bar{x}_1 + A_2 \bar{x}_2$$

$$A_1 = \frac{2PL^2}{EI}$$

$$\bar{x}_1 = \frac{4}{3}L$$

$$A_2 = -\frac{k}{2}\frac{PL^2}{EI}$$

$$\bar{x}_2 = L + \frac{2}{3}L = \frac{5}{3}L.$$

$$\Rightarrow \Delta_c = \underbrace{\frac{2PL^2}{EI}}_{A_1} \cdot \underbrace{\frac{4}{3}L}_{\bar{x}_1} - \underbrace{\frac{k}{2}\frac{PL^2}{EI}}_{A_2} \cdot \underbrace{\frac{5}{3}L}_{\bar{x}_2}$$

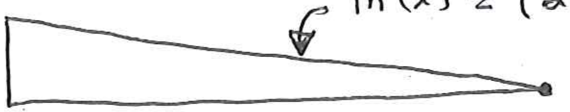
$$= \frac{PL^3}{3EI}\left[8 - \frac{5k}{2}\right].$$

[1f] (5 pts) Use the method of <u>**VIRTUAL FORCES**</u> to verify that the vertical deflection of point C is:

$$\Delta_C = \frac{PL^3}{3EI}\left[8 - \frac{5k}{2}\right]. \qquad (3)$$

Apply unit force at C

$m(x) = (2L - x)$



From previous section

$$M_1(x) = P(2L - x), \qquad 0 \le x \le 2L$$

$$M_2(x) = -Pk(L - x), \qquad 0 \le x \le L$$

$$\Delta_c = \underbrace{\int_0^{2L} \frac{M_1(x)}{EI} \cdot m(x)\,dx}_{I_1} - \underbrace{\int_0^{L} \frac{M_2(x)}{EI}\, m(x)\,dx}_{I_2}.$$

$$I_1 = \int_0^{2L} \frac{P(2L-x)}{EI}(2L - x)\,dx = \frac{8}{3}\frac{PL^3}{EI}$$

$$I_2 = \int_0^{L} \frac{Pk(L-x)}{EI}(2L - x)\,dx = \frac{5}{6}\frac{PkL^3}{EI}.$$

$$\Rightarrow \Delta_c = I_1 - I_2 = \frac{PL^3}{3EI}\left[8 - \frac{5k}{2}\right].$$

Question 2: 10 points

OPTIONAL: Structural Analysis of a Simple Beam Structure. The beam structure shown in Figure 2 supports an external load P at point B.

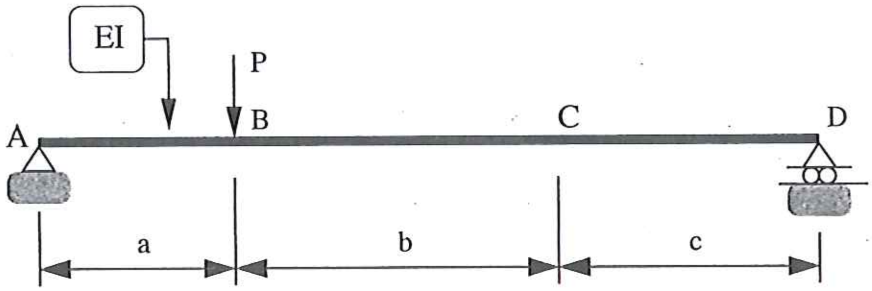


Figure 2: Front elevation view of a simple beam structure.

[2a] (4 pts) Use the method of **VIRTUAL DISPLACEMENTS** to compute formulae for the vertical reactions at A and D. Show all of your working.

$$\Delta_B^{**} = \frac{a}{a+b+c}\,\Delta^{**}$$

EWD = 0

$$V_D\,\Delta^{**} + \Delta_B^{**} \cdot (-P) = 0$$

$$\Rightarrow V_D = \left(\frac{a}{a+b+c}\right)P.$$

$$\Delta_B^{**} = \frac{b+c}{a+b+c}\,\Delta^{**}$$

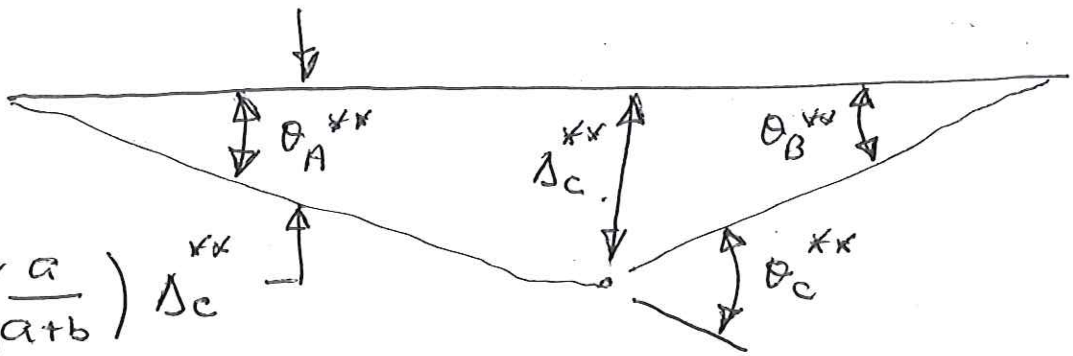
$$V_A \cdot \Delta^{**} + \Delta_B \cdot (-P) = 0$$

$$\Rightarrow V_A = \left(\frac{b+c}{a+b+c}\right)P$$

Note: $V_A + V_B = P$ ✓.

[2b] (6 pts) Use the method of **VIRTUAL DISPLACEMENTS** to compute a formula for the bending moment at C. Show all of your working.

Apply virtual rotation at C.



$$\Delta_B^{**} = \left(\frac{a}{a+b}\right) \Delta_c^{**}$$

$\sum' EWD = 0$

$$\Rightarrow M_c \cdot \theta_c^{**} = P\,\Delta_B^{**} \qquad \text{—— (A)}$$

From geometry. $\theta_c^{**} = \theta_A^{**} + \theta_B^{**}$ —— (B)

Also $\Delta_c = \theta_A^{**}(a+b) = \theta_B^{**} \cdot c$ — (C)

Plug (B) & (C) into (A).

$$M_c\,(\theta_A^{**} + \theta_B^{**}) = P\left(\frac{a}{a+b}\right)\Delta_c^{**}$$

$$\Rightarrow \quad M_c\left[\frac{1}{c} + \frac{1}{a+b}\right]\Delta_c^{**} = P\left[\frac{a}{a+b}\right]\Delta_c^{**}$$

$$\Rightarrow M_c = \left\{\frac{ac(a+b)}{(a+b+c)(a+b)}\right] P, = \left[\frac{ac}{a+b+c}\right] P.$$

$= V_D \cdot c \leftarrow$ by statics.

Question 3: 10 points

OPTIONAL: Simple Three-Pinned Arch. Figure 3 is a front elevation view of a simple three-pinned arch. Horizontal and vertical loads P are applied at nodes C and E, respectively.

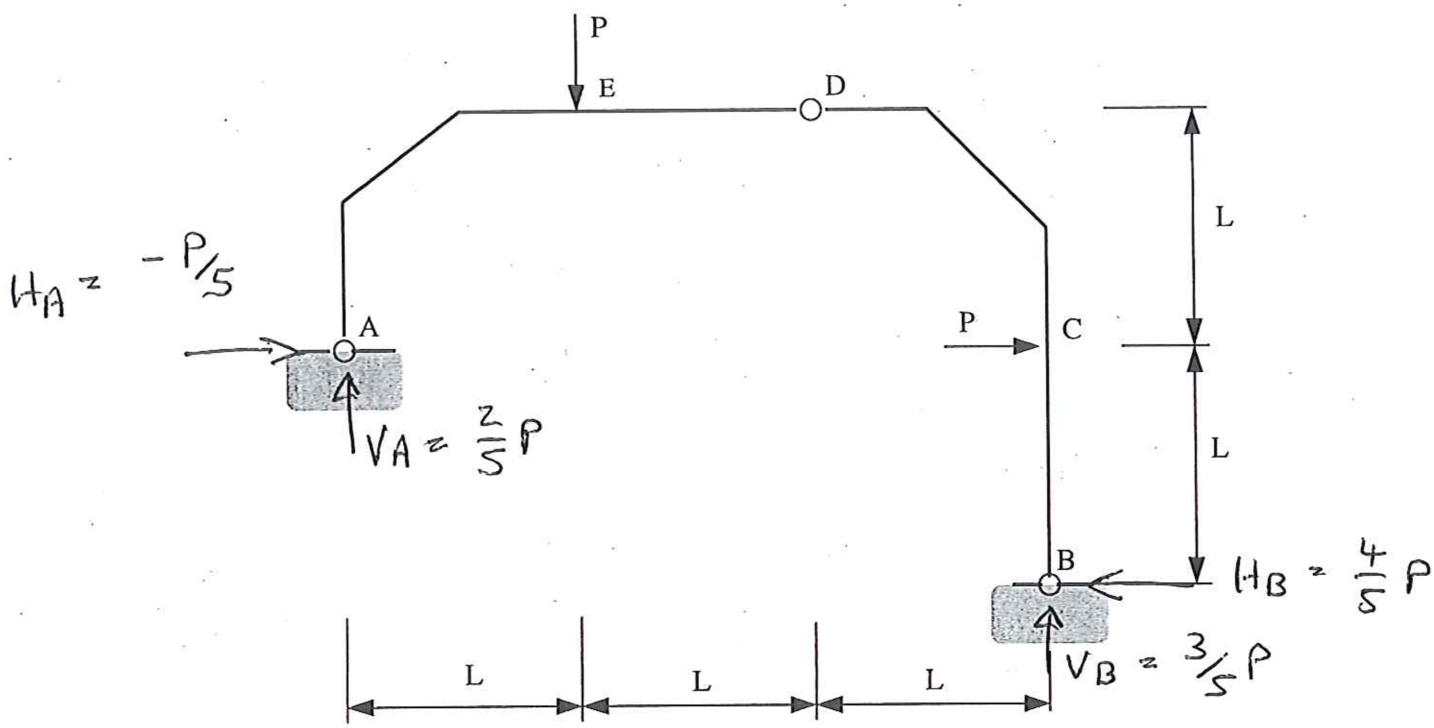


Figure 3: Front elevation view of a simple three-pinned arch.

[3a] (4 pts) Compute the vertical and horizontal components of reaction force at supports A and B as a function of P.

$$\sum V = 0 \Rightarrow V_A + V_B = P \quad - (A)$$
$$\sum H = 0 \Rightarrow H_A + P = H_B \quad - (B)$$
$$\sum M_D = 0 \Rightarrow V_B + P = \cancel{2V_A}\ 2H_B \quad - (C)$$
$$\sum M_D = 0 \Rightarrow \cancel{V} H_A + P = 2V_A \quad - (D)$$

From (B) & (D)

$$H_B = 2V_A$$

From (C) & (A)

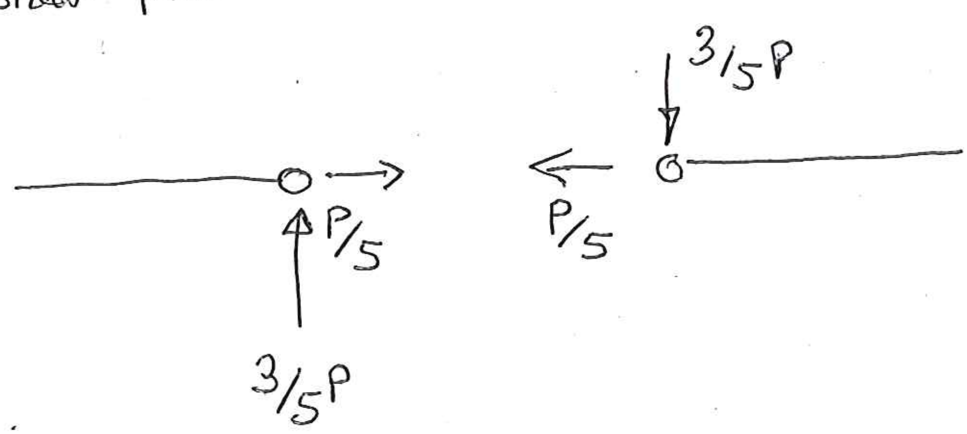
$$V_B + P = 4V_A$$
$$V_A + V_B = P$$

$$\Rightarrow V_B + P = 4(P - V_B) \Rightarrow V_B = \frac{3}{5}P \qquad H_A = \frac{-P}{5}$$
$$V_A = \frac{2}{5}P \qquad H_B = 4/5\,P.$$

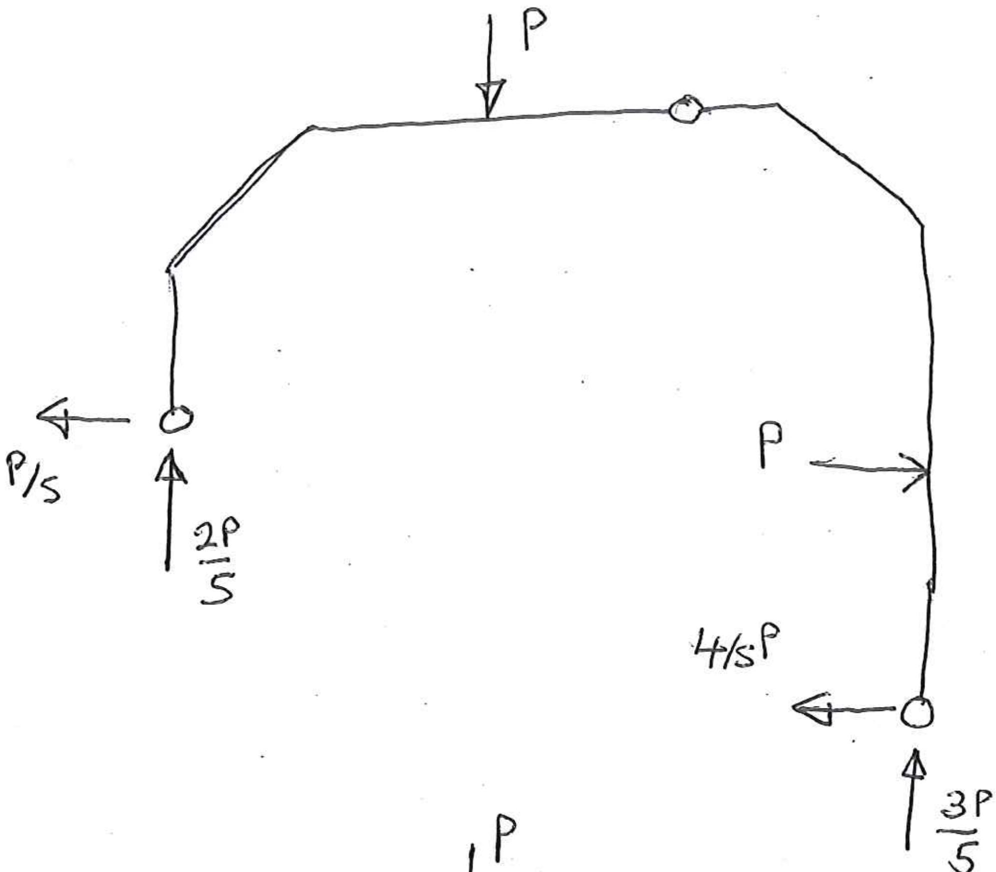
Queation 3a continued:

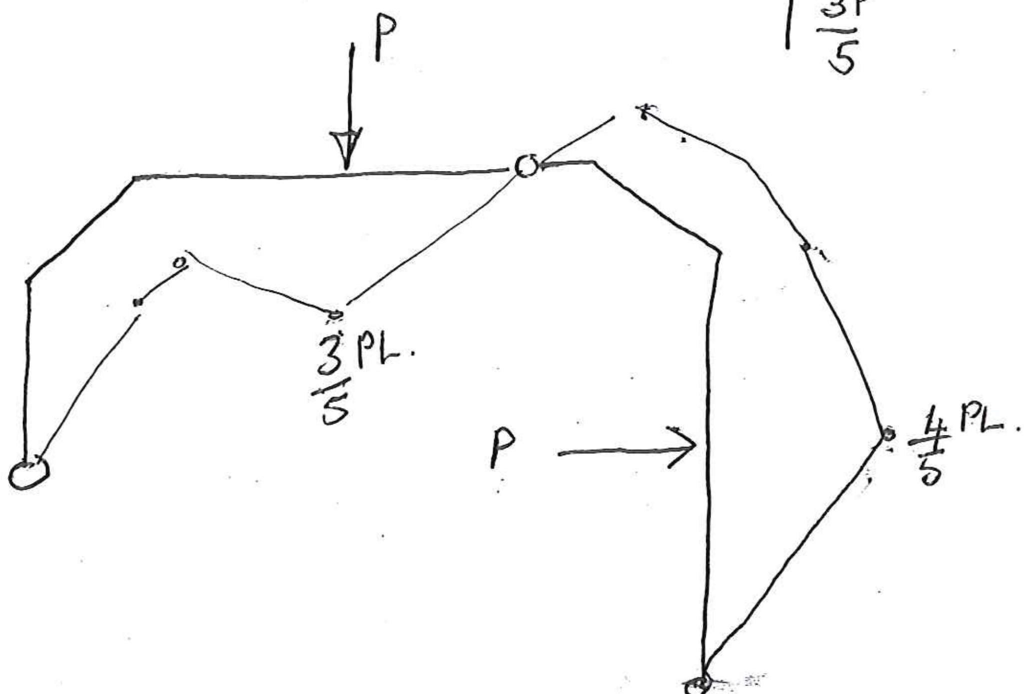
[3b] (3 pts) Compute and axial and shear forces transfered across the hinge at D. You can annotate Figure 3 if you think it will help to explain your solution.

Axial & Shear forces at D



[3c] (3 pts) Draw the bending moment diagram.





Question 4: 10 points

OPTIONAL: Principle of Virtual Work. Figure 4 is a front elevation view of a simple truss that supports vertical loads at nodes C and D. All of the truss members have cross section properties AE.

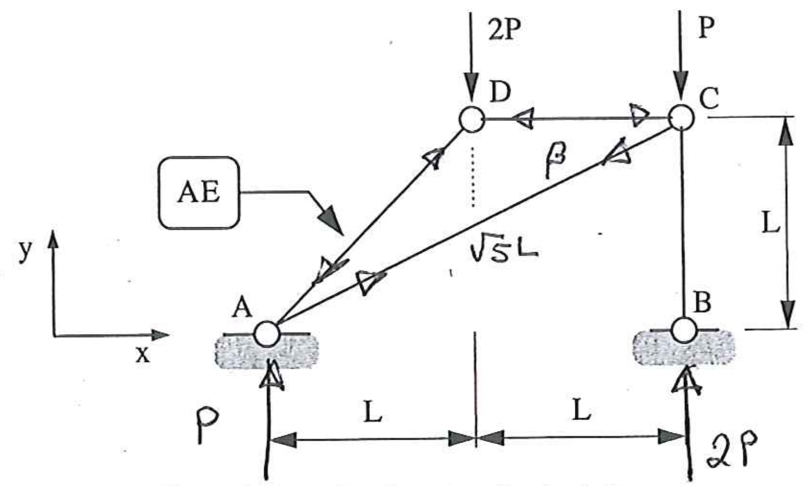


Figure 4: Front elevation view of a simple truss.

[4a] (5 pts). Compute the support reactions and distribution of forces throughout the structure.

$\sum M_A = 0 \qquad 2PL + 2PL = 2L \cdot \overline{CB}$

$\Rightarrow \quad CB = -2P \ (C).$

$\sum V_C = 0 \qquad \dfrac{\overline{CA}}{\sqrt{5}} \cdot = P \quad \Rightarrow \quad \overline{CA} = \sqrt{5}P \ (T).$

$\sum H_C = 0 \qquad \overline{CD} \pm \overline{CA} \cos\beta = 0$

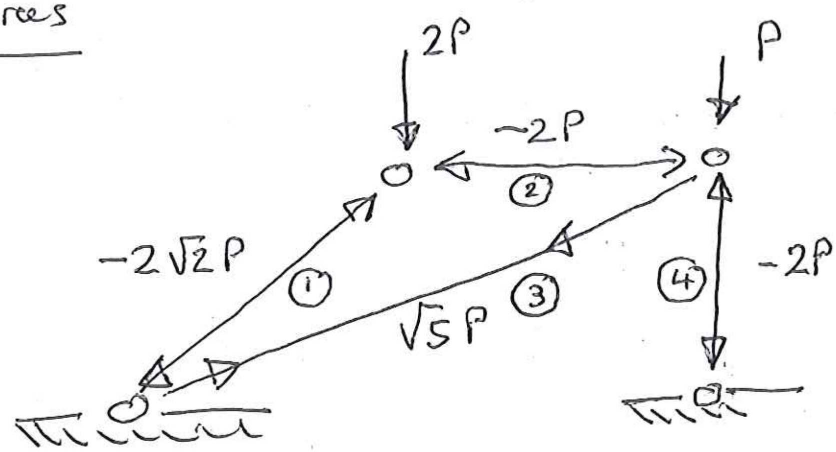
$\Rightarrow \quad \overline{CD} = -\dfrac{2}{\sqrt{5}} \cdot \sqrt{5}P = -2P \ (C)$

$\sum V_D = 0 \quad \Rightarrow \quad \overline{AD} = -2\sqrt{2}P \ (C).$

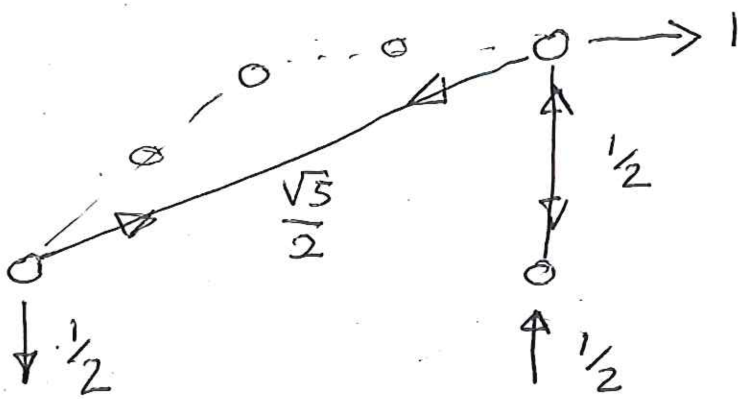
[4b] (5 pts). Use the method of **VIRTUAL FORCES** to show that the horizontal deflection at node C is:

$$\triangle = \frac{PL}{AE}\left[1+\frac{5\sqrt{5}}{2}\right]. \qquad (4)$$

Real forces



Unit load at C



#	L/AE	F	f_1	$\frac{F f_1 L}{AE}$
1	$\sqrt{2}L/AE$	$-2\sqrt{2}P$	0	0
2	L/AE	$-2P$	0	0
3	$\sqrt{5}L/AE$	$\sqrt{5}P$	$\sqrt{5}/2$	$\frac{5\sqrt{5}}{2}\frac{PL}{AE}$
4	L/AE	$-2P$	$-1/2$	$\frac{PL}{AE}$
				$\left(1+\frac{5\sqrt{5}}{2}\right)PL/AE$

Horizontal displ

$$= \left(1+\frac{5\sqrt{5}}{2}\right)\frac{PL}{AE}$$

Question 5: 10 points

OPTIONAL: Deriving flexibility matrices for a simple beam. Consider the simply supported beam structure shown in Figure 5.

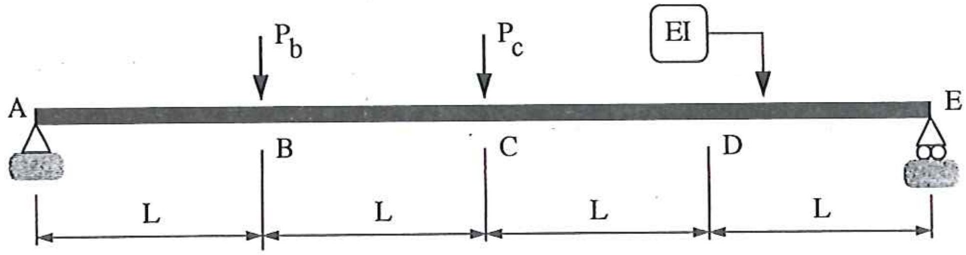
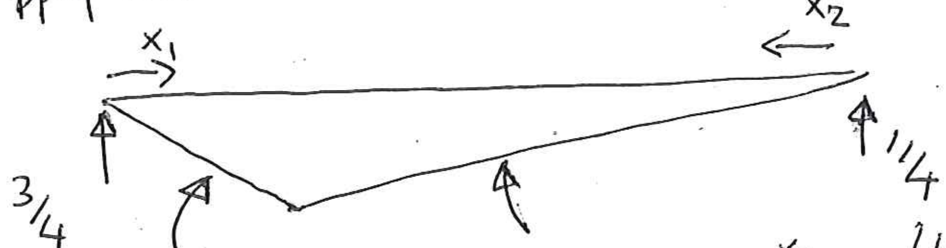


Figure 5: Front elevation view of a simply supported beam.

[5a] (10 pts) Use the method of **VIRTUAL FORCES** to compute the two-by-two flexibility matrix connecting the vertical displacements at points B and C to applied loads P_b and P_c, i.e.,

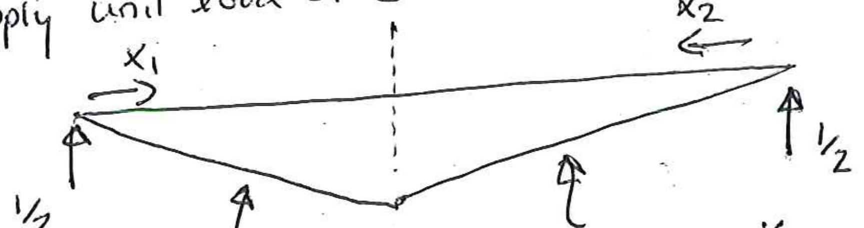
$$\begin{bmatrix} \Delta_b \\ \Delta_c \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} P_b \\ P_c \end{bmatrix}. \tag{5}$$

Apply unit load at B



$M_1(x) = \frac{3}{4} x_1$ $\qquad$ $M_1(x_2) = \frac{x_2}{4} = \frac{(4L - x_1)}{4}.$

Apply unit load at C



$M_2(x) = x_1/2$ $\qquad$ $M_2(x) = \frac{x_2}{2} = \left(\frac{4L - x_1}{2}\right)$

← Axis of symmetry.

Question 5 continued ...

$$f_{11} = \frac{1}{EI}\int_0^L \left(\frac{3}{4}x_1\right)^2 dx_1 + \frac{1}{EI}\int_0^{3L}\left(\frac{x_2}{4}\right)^2 \cdot dx_2 = \frac{3}{4}\frac{L^3}{EI}$$

$$f_{22} = 2\int_0^{2L} \frac{M_1^2(x)}{EI}\,dx = \frac{2}{EI}\int_0^{2L}\left(\frac{x}{2}\right)^2 dx = \frac{8L^3}{6EI}$$

$$f_{12} = \underbrace{\frac{1}{EI}\int_0^L \frac{3}{4}x \cdot \frac{1}{2}x\,dx}_{I_1} + \underbrace{\int_L^{2L} \frac{1}{EI}\frac{x}{2}\left(\frac{4L-x}{4}\right)dx}_{I_2} + \underbrace{\int_0^{2L}\frac{1}{EI}\left(\frac{x}{2}\right)\left(\frac{x}{4}\right)dx}_{I_3}$$

$$I_1 = \frac{3}{8EI}\left[\frac{x^3}{3}\right]_0^L = \frac{L^3}{8EI}.$$

$$I_2 = \int_L^{2L} \frac{1}{8EI}(4Lx - x^2)\,dx = \frac{1}{8EI}\left[2Lx^2 - \frac{x^3}{3}\right]_L^{2L} = \frac{11}{24}\frac{L^3}{EI}$$

$$I_3 = \frac{1}{8EI}\int_0^{2L} x^2\,dx = \frac{1}{8EI}\left[\frac{x^3}{3}\right]_0^{2L} = \frac{L^3}{3EI}$$

$$f_{12} = I_1 + I_2 + I_3 = \frac{L^3}{EI}\left[\frac{1}{8} + \frac{11}{24} + \frac{1}{3}\right] = \frac{11}{12}\frac{L^3}{EI}.$$

$$f = \begin{bmatrix} \frac{3}{4}L^3/EI & \frac{11}{12}L^3/EI \\ \frac{11}{12}L^3/EI & \frac{4}{3}L^3/EI \end{bmatrix} = \frac{L^3}{12EI}\begin{bmatrix} 9 & 11 \\ 11 & 16 \end{bmatrix}$$

Question 6: 10 points

OPTIONAL: Compute Support Reactions in a Suspension Bridge. Figure 6 is a front elevation view of a suspension bridge that has three spans, two support towers, and anchor supports at the bridge endpoints.

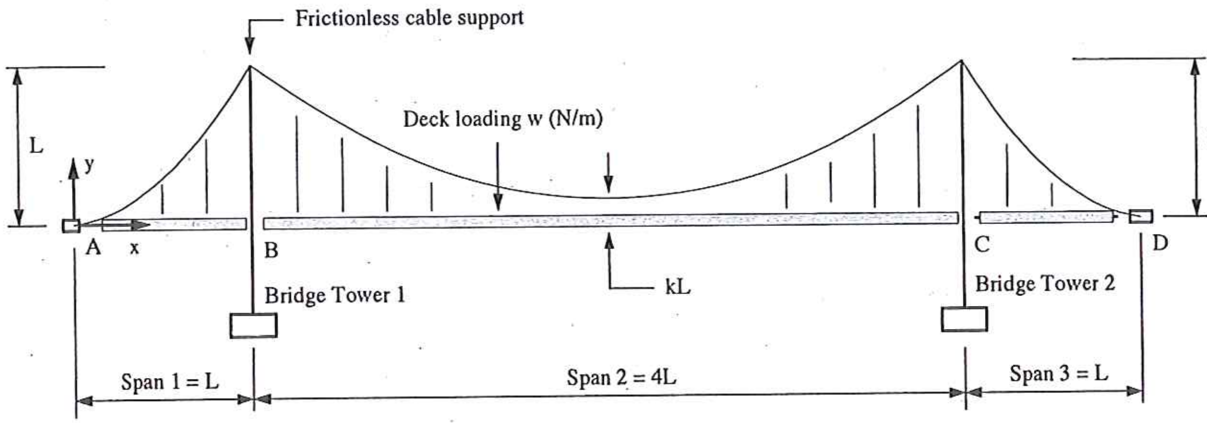


Figure 6: Elevation view of a three-span suspension bridge.

Spans 1, 2 and 3 have lengths L, $4L$ and L, respectively, and carry a uniform load w (N/m). The towers have height L above the bridge deck level. At the mid-point of Span 2, the lowest point of the cable profile is kL above the bridge deck.

The purpose of this question is to work step by step toward the computation of vertical support reactions at the cable anchor points (i.e., at points A and D), and at the towers (i.e., at points B and C). For the purposes of this analysis, assume that the bridge cable weight is bundled into the applied loads w, and can otherwise be ignored. Also, assume that the cable passes through the top of the towers on a frictionless support and, as a result, the horizontal component of cable force will be constant along the entire bridge.

[6a] (2 pts) By examining equilibrium of the cable profile in Span 2, show that the horizontal component of cable force is:

$$H = \left[\frac{2}{1-k}\right] wL. \tag{6}$$

$$\frac{d^2y}{dx^2} = \frac{w}{H}$$

$$y(x) = \frac{wx^2}{2H}$$

$$y(2L) = (1-k)L \rightarrow H = \left[\frac{2}{1-k}\right] wL.$$

$$= \frac{w\,4L^2}{2H} \Rightarrow (1-k)L = \frac{2L^2 w}{H}$$

[6b] (3 pts) Show that the equation of the cable profile in Span 1 is:

$$y(x) = \left[\frac{1-k}{4L}\right]x^2 + \left[\frac{3+k}{4}\right]x \tag{7}$$

$$\frac{d^2y}{dx^2} = \frac{W}{H}$$

$$\frac{dy}{dx} = \frac{Wx}{H} + A$$

$$y(x) = \frac{Wx^2}{2H} + Ax + B$$

$$\left.\begin{array}{l} y(0) = 0 \\ y(L) = L \end{array}\right\} \Rightarrow \begin{array}{l} B = 0 \\ A = \left(1 - \frac{WL}{2H}\right) \end{array}$$

$$y(x) = \frac{Wx^2}{2H} + \left(1 - \frac{WL}{2H}\right)x$$

but $H = \left(\frac{2}{1-k}\right)WL$

$$\Rightarrow y(x) = \left(\frac{1-k}{4L}\right)x^2 + \left(\frac{3+k}{4}\right)x.$$

[6c] (3 pts) Use the results from parts [6a] and [6b] to show that the vertical reaction at the anchor support (i.e., at point A) is:

$$V_A = \left[\frac{3+k}{2-2k}\right]wL \tag{8}$$

acting downwards.

$$\left.\frac{dy}{dx}\right|_{x=0} = \left[\frac{Wx}{H} + 1 - \frac{WL}{2H}\right]_{x=0} = \frac{V}{H}$$

$$H = \left(\frac{2}{1-k}\right)WL \Rightarrow V_A = \left(\frac{3+k}{2-2k}\right)WL.$$

[6d] (2 pts) Hence, show that the vertical support reaction at Tower 1 is:

$$V_B = \left[\frac{9-5k}{2-2k}\right] wL. \tag{9}$$

acting upwards.

$$V_B = 3wL + V_A = \left(\frac{6-6k}{2-2k}\right) wL + \frac{(3+k)}{(2-2k)} wL$$

$$= \left(\frac{9-5k}{2-2k}\right) wL.$$

BONUS PROBLEM: 5 points

One of the themes of this semester has been that symmetries simplify structural analysis. With that in mind (hint, hint, hint), here is a math problem that looks very intimidating, but can be solved with middle school math (no derivatives, no calculus!).

Problem: A triangle is constructed by connecting the three vertices shown in Figure 7.

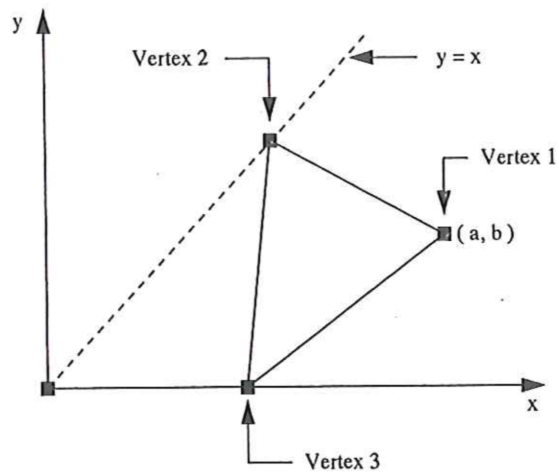


Figure 7: Schematic of a triangle having vertices 1 through 3.

Vertex 1 is fixed at coordinate (a,b) and cannot be moved. Vertices 2 and 3 are attached to points somewhere along the lines y = x, and the x axis, respectively. Prove that the minimum value of perimeter for the triangle is:

$$\sqrt{2}\sqrt{a^2+b^2} \tag{10}$$

Justify your solution with the appropriate equations and rationale.