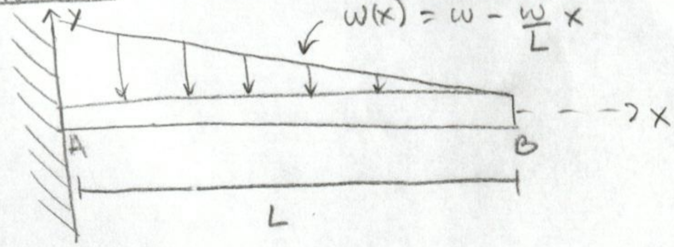
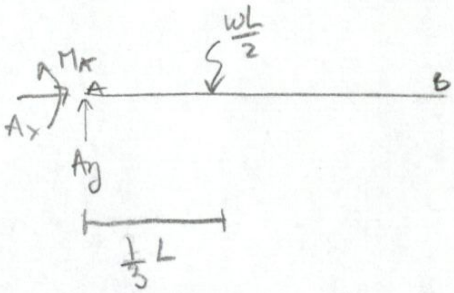
In-Class Problem #4:



a) Derive the equation of the elastic curve $y(x)$ in terms of w, L, x, E and I.

Step 1: Solve for reactions



$\Sigma F_x = 0$

$A_x = 0$

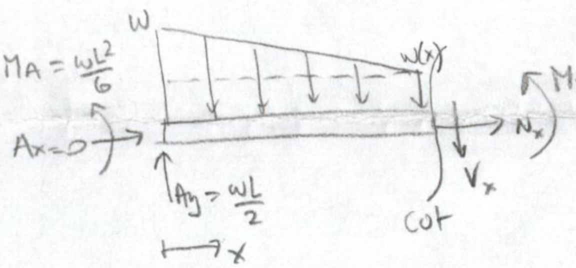
$\Sigma M_A = 0$

$M_A = \frac{wL}{2}\left(\frac{1}{3}L\right) = \frac{wL^2}{6}$

$\Sigma F_y = 0$

$A_y = \frac{wL}{2}$

Step 2: Solve for $M(x)$



$+\circlearrowleft \Sigma M_x = 0$

$M_x + M_A - A_y(x) + \left[\frac{w - w(x)}{2}\right](x)\left(\frac{2}{3}x\right) + w(x)(x)\left(\frac{x}{2}\right) = 0$

$M_x + \frac{wL^2}{6} - \frac{wL}{2}(x) + \left[\frac{w - \left(w - \frac{w}{L}x\right)}{2}\right](x)\left(\frac{2}{3}x\right) + \left(w - \frac{w}{L}x\right)(x)\left(\frac{x}{2}\right) = 0$

$M_x + \frac{wL^2}{6} - \frac{wL}{2}x + \cancel{\frac{wx^2}{3}} - \cancel{\frac{wx^2}{3}} + \frac{wx^3}{3L} + \frac{wx^2}{2} - \frac{wx^3}{2L} = 0$

$M_x + \frac{wL^2}{6} - \frac{wL}{2}x + \frac{wx^2}{2} - \frac{wx^3}{6L} = 0$

$M_x = -\frac{wL^2}{6} + \frac{wL}{2}x - \frac{wx^2}{2} + \frac{wx^3}{6L}$

Step 3: Determine $y(x)$

$\frac{d^2y}{dx^2} = \frac{1}{EI}M(x)$

$\frac{dy}{dx} = \frac{1}{EI}\int -\frac{wL^2}{6} + \frac{wL}{2}x - \frac{wx^2}{2} + \frac{wx^3}{6L}\, dx$

$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{wL^2}{6}x + \frac{wL}{4}x^2 - \frac{wx^3}{6} + \frac{wx^4}{24L} + C_1\right]$

$y(x) = \frac{1}{EI}\int -\frac{wL^2}{6}x + \frac{wL}{4}x^2 - \frac{wx^3}{6} + \frac{wx^4}{24L} + C_1\, dx$

$y(x) = \frac{1}{EI}\left[-\frac{wL^2}{12}x^2 + \frac{wL}{12}x^3 - \frac{wx^4}{24} + \frac{wx^5}{120L} + C_1x + C_2\right]$

Boundary conditions: $\frac{dy}{dx}$ @ $x=0$ is zero

$y(x)$ @ $x=0$ is zero

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{1}{EI} c_1 = 0$$

$$c_1 = 0$$

$$y(0) = \frac{1}{EI} c_2 = 0$$

$$c_2 = 0$$

$$\boxed{y(x) = \frac{1}{EI}\left[-\frac{wL^2}{12}x^2 + \frac{wL}{12}x^3 - \frac{wx^4}{24} + \frac{wx^5}{120L}\right]}$$

b) Determine the slope at the right end of the beam

$$\left.\frac{dy}{dx}\right|_{x=L} = \frac{1}{EI}\left[-\frac{wL^2}{6}(L) + \frac{wL}{4}(L)^2 - \frac{w(L)^3}{6} + \frac{w(L)^4}{24L}\right]$$

$$\left.\frac{dy}{dx}\right|_{x=L} = \frac{1}{EI}\left[-\frac{wL^3}{6} + \frac{wL^3}{4} - \frac{wL^3}{6} + \frac{wL^3}{24}\right]$$

$$\boxed{\left.\frac{dy}{dx}\right|_{x=L} = -\frac{wL^3}{24}}$$

c) Determine the deflection at $x=L$.

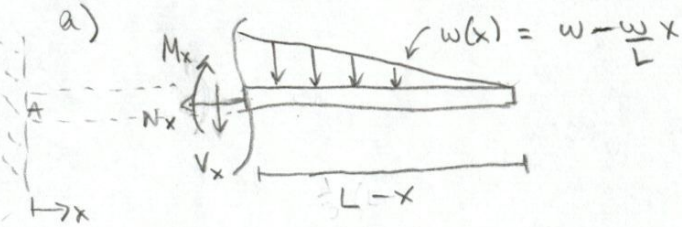
$$y(L) = \frac{1}{EI}\left[-\frac{wL^2}{12}(L)^2 + \frac{wL}{12}(L)^3 - \frac{w(L)^4}{24} + \frac{w(L)^5}{120L}\right]$$

$$y(L) = \frac{1}{EI}\left[\cancel{-\frac{wL^4}{12}} + \cancel{\frac{wL^4}{12}} - \frac{wL^4}{24} + \frac{wL^4}{120}\right]$$

$$\boxed{y(L) = -\frac{wL^4}{30}}$$

OR

a)



$$\circlearrowleft + \sum M_x = -\frac{w(x)(L-x)}{2}\left(\frac{1}{3}\right)(L-x) - M_x = 0$$

$$M_x = -\left(w - \frac{w}{L}x\right)\frac{(L-x)^2}{6}$$

$$M_x = -\frac{w(L-x)^2}{6} + \frac{wx(L-x)^2}{6L}$$

$$M_x = -\frac{w(L^2 - 2xL + x^2)}{6} + \frac{wx(L^2 - 2xL + x^2)}{6L}$$

$$M_x = -\frac{wL^2}{6} + \frac{2xLw}{6} - \frac{wx^2}{6} + \frac{wxL}{6} - \frac{2x^2w}{6} + \frac{wx^3}{6L}$$

$$M_x = -\frac{wL^2}{6} + \frac{wLx}{2} - \frac{wx^2}{2} + \frac{wx^3}{6L}$$

$$\frac{d^2y}{dx^2} = \frac{1}{EI}M(x)$$

$$\frac{dy}{dx} = \frac{1}{EI}\int -\frac{wL^2}{6} + \frac{wLx}{2} - \frac{wx^2}{2} + \frac{wx^3}{6L}\, dx$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{wL^2}{6}x + \frac{wL}{4}x^2 - \frac{wx^3}{6} + \frac{wx^4}{24L} + c_1\right]$$

$$y(x) = \frac{1}{EI}\int -\frac{wL^2}{6}x + \frac{wL}{4}x^2 - \frac{wx^3}{6} + \frac{wx^4}{24L} + c_1\, dx$$

$$y(x) = \frac{1}{EI}\left[-\frac{wL^2}{12}x^2 + \frac{wLx^3}{12} - \frac{wx^4}{24} + \frac{wx^5}{120L} + c_1x + c_2\right]$$

Boundary conditions: $\left.\frac{dy}{dx}\right|_{x=0} = \frac{1}{EI}c_1 = 0 \rightarrow c_1 = 0$

$$y(0) = \frac{1}{EI}c_2 = 0 \rightarrow c_2 = 0$$

$$\boxed{y(x) = \frac{1}{EI}\left[-\frac{wL^2x^2}{12} + \frac{wLx^3}{12} - \frac{wx^4}{24} + \frac{wx^5}{120L}\right]}$$

b) Determine the slope at the right end of the beam

$$\left.\frac{dy}{dx}\right|_{x=L} = \frac{1}{EI}\left[-\frac{wL^2}{6}(L) + \frac{wL}{4}(L)^2 - \frac{w(L)^3}{6} + \frac{w(L)^4}{24L}\right]$$

$$\left.\frac{dy}{dx}\right|_{x=L} = \frac{1}{EI}\left[-\frac{wL^3}{6} + \frac{wL^3}{4} - \frac{wL^3}{6} + \frac{wL^3}{24}\right]$$

$$\left.\frac{dy}{dx}\right|_{x=L} = -\frac{wL^3}{24}$$

c) Determine the deflection @ $x = L$

$$y(L) = \frac{1}{EI}\left[-\frac{wL^2}{12}(L)^2 + \frac{wL(L)^3}{12} - \frac{w(L)^4}{24} + \frac{w(L)^4}{120}\right]$$

$$y(L) = \frac{1}{EI}\left[-\frac{wL^4}{12} + \frac{wL^4}{12} - \frac{wL^4}{24} + \frac{wL^4}{120}\right]$$

$$\boxed{y(L) = -\frac{wL^4}{30}}$$