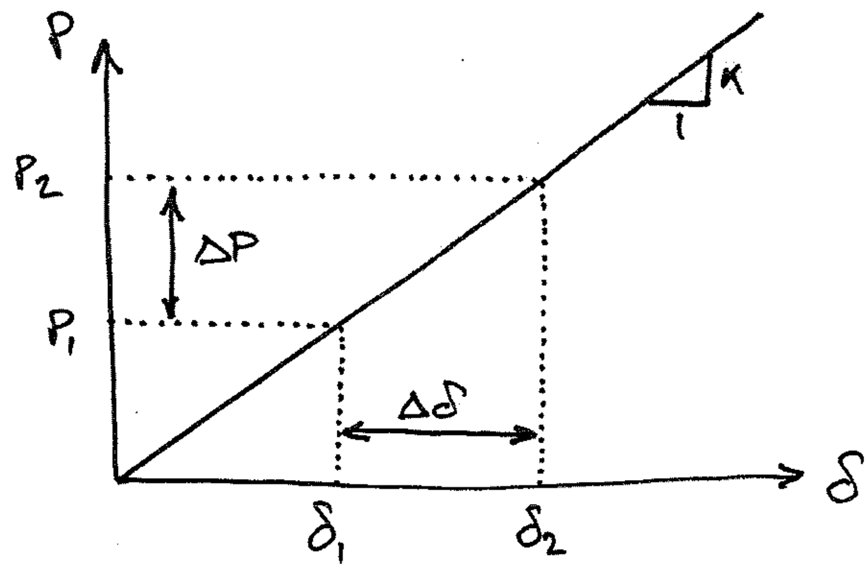
3. Principle of Superposition

3.1 Development

For a linearly elastic structure, load, P, and deformation, δ, are related through stiffness, K, as shown:



For an initial load on the structure we have:

$$P_1 = K \cdot \delta_1$$

If we instead we had applied ΔP we would have gotten:

$$\Delta P = K \cdot \Delta\delta$$

Now instead of applying ΔP separately to P_1 we apply it after P_1 is already applied. The final forces and deflections are got by adding the equations:

$$\begin{aligned} P_1 + \Delta P &= K \cdot \delta_1 + K \cdot \Delta\delta \\ &= K\left(\delta_1 + \Delta\delta\right) \end{aligned}$$

But, since from the diagram, $P_2 = P_1 + \Delta P$ and $\delta_2 = \delta_1 + \Delta\delta$, we have:

$$P_2 = K \cdot \delta_2$$

which is a result we expected.

This result, though again deceptively 'obvious', tells us that:

- Deflection caused by a force can be added to the deflection caused by another force to get the deflection resulting from both forces being applied;
- The order of loading is not important (ΔP or P_1 could be first);
- Loads and their resulting load effects can be added or subtracted for a structure.

This is the **Principle of Superposition**:

For a linearly elastic structure, the load effects caused by two or more loadings are the sum of the load effects caused by each loading separately.

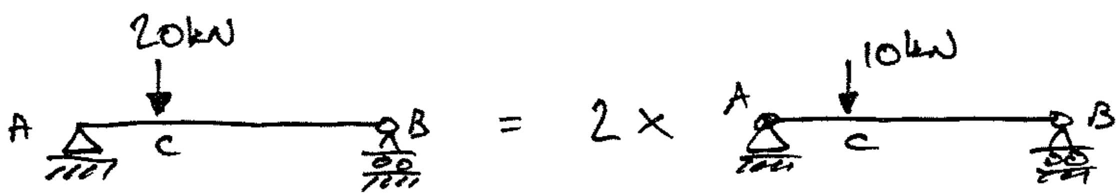
Note that the principle is limited to:

- Linear material behaviour only;
- Structures undergoing small deformations only (linear geometry).

3.2 *Example*

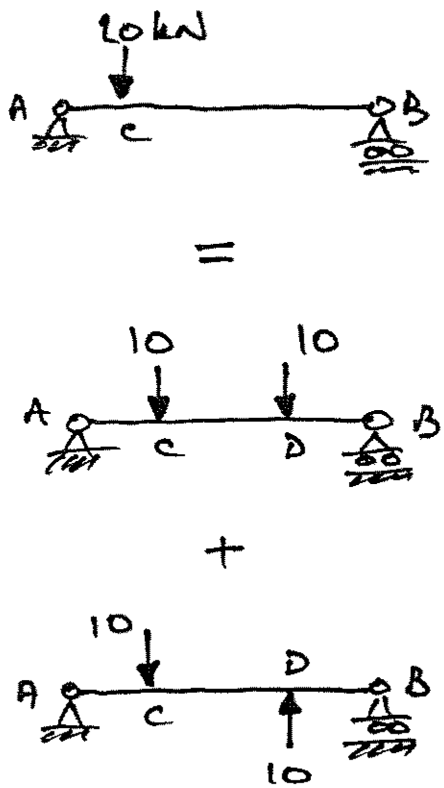
If we take a simply-supported beam, we can see that its solutions can be arrived at by multiplying the solution of another beam:

Ex 1



The above is quite obvious, but not so obvious is that we can also break the beam up as follows:

Ex 2



Thus the principle is very flexible and useful in solving structures.

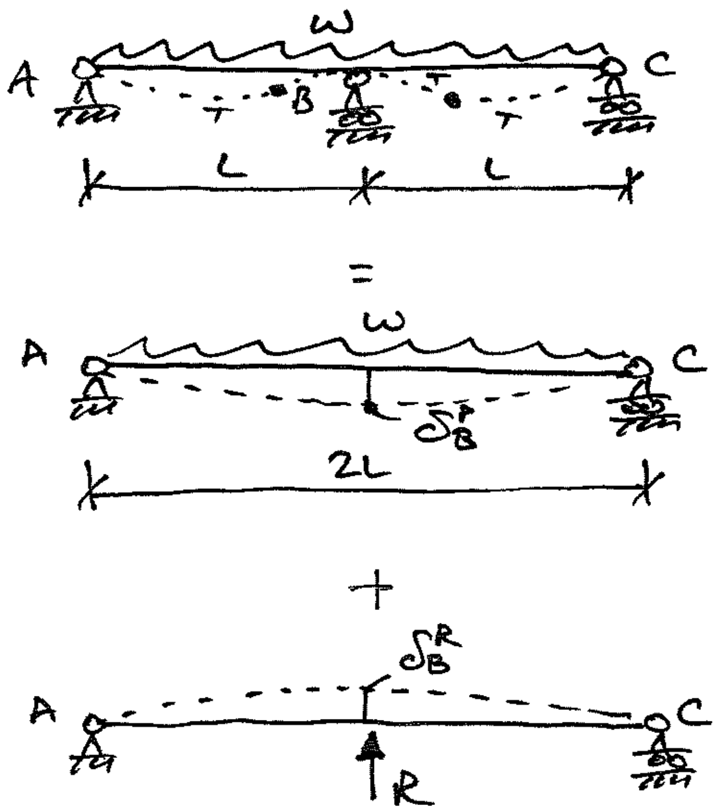
4. Solving Indeterminate Structures

4.1 Introduction

Compatibility of displacement along with superposition enables us to solve indeterminate structures. Though we'll use more specialized techniques they will be fundamentally based upon the preceding ideas. Some simple example applications follow.

4.3 Example: 2-Span Beam

Considering a 2-span beam, subject to UDL, which has equal spans, we break it up using the principle of superposition:



Once again we use compatibility of displacements for the original structure to write:

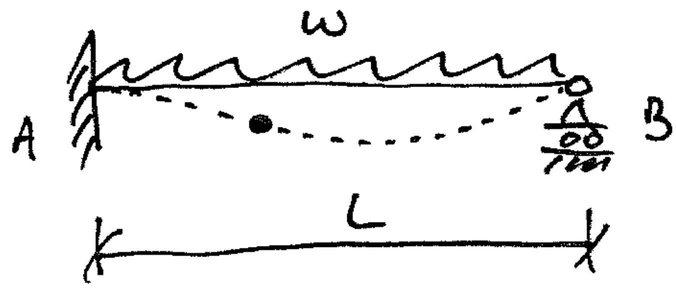
$$\delta_B = \delta_B^P + \delta_B^R = 0$$

Again, from tables of standard deflections, we have:

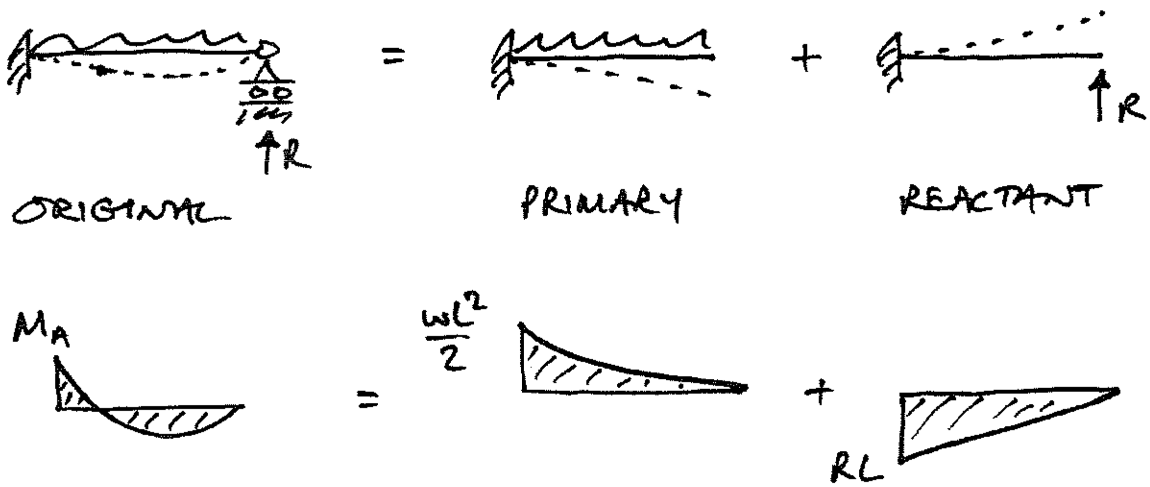
$$\delta_B^P = +\frac{5w(2L)^4}{384EI} = +\frac{80wL^4}{384EI}$$

4.2 Example: Propped Cantilever

Consider the following propped cantilever subject to UDL:



Using superposition we can break it up as follows (i.e. we choose a redundant):



Next, we consider the deflections of the primary and reactant structures:

Now by compatibility of displacements for the original structure, we know that we need to have a final deflection of zero after adding the primary and reactant deflections at *B*:

$$\delta_B = \delta_B^P + \delta_B^R = 0$$

From tables of standard deflections, we have:

$$\delta_B^P = +\frac{wL^4}{8EI} \text{ and } \delta_B^R = -\frac{RL^3}{3EI}$$

In which downwards deflections are taken as positive. Thus we have:

$$\begin{aligned} \delta_B &= +\frac{wL^4}{8EI} - \frac{RL^3}{3EI} = 0 \\ \therefore R &= \frac{3wL}{8} \end{aligned}$$

Knowing this, we can now solve for any other load effect. For example:

$$\begin{aligned} M_A &= \frac{wL^2}{2} - RL \\ &= \frac{wL^2}{2} - \frac{3wL}{8}L \\ &= \frac{4wL^2 - 3wL^2}{8} \\ &= \frac{wL^2}{8} \end{aligned}$$

Note that the $wL^2/8$ term arises without a simply-supported beam in sight!

And:

$$\delta_B^R = -\frac{R(2L)^3}{48EI} = -\frac{8RL^3}{48EI}$$

In which downwards deflections are taken as positive. Thus we have:

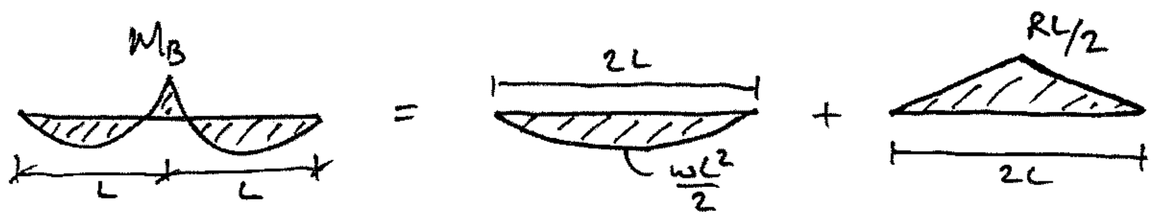
$$\delta_B = +\frac{80wL^4}{384EI} - \frac{8RL^3}{48EI} = 0$$

$$\frac{8R}{48} = \frac{80wL}{384}$$

$$R = \frac{10wL}{8}$$

Note that this is conventionally not reduced to $5wL/4$ since the other reactions are both $3wL/8$. *Show this as an exercise.*

Further, the moment at B is by superposition:



Hence:

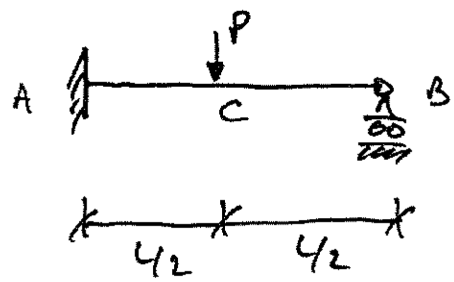
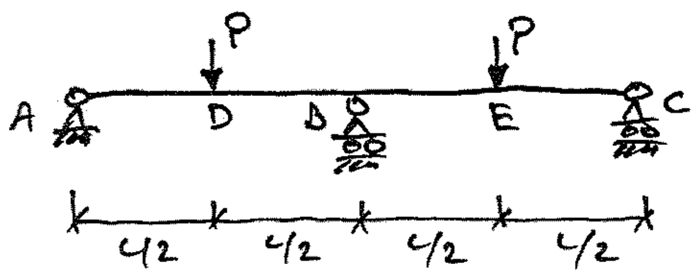
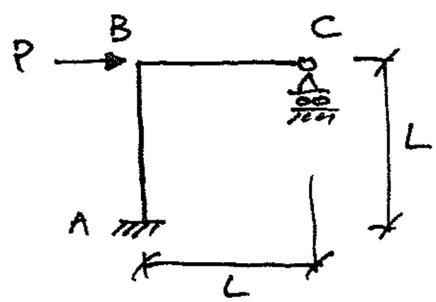
$$M_B = \frac{RL}{2} - \frac{wL^2}{2} = \frac{10wL}{8}\cdot\frac{L}{2} - \frac{wL^2}{2} = \frac{10wL^2 - 8wL^2}{16}$$

$$= \frac{wL^2}{8}$$

And again $wL^2/8$ arises!

5. Problems

Use compatibility of displacement and the principle of superposition to solve the following structures. In each case draw the bending moment diagram and determine the reactions.

1.	
2.	
3.	 This one is tricky: choosing the reaction at C gives $R = 3P/8$.