Roots of Equations

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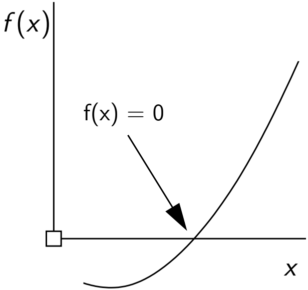
Overview

Part 1

Numerical Solution of Equations

Numerical Solution of Equations

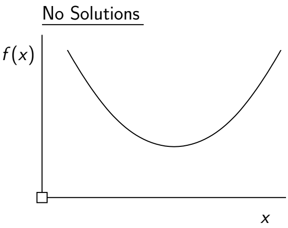
Math Problem. Given f(x), find a value of x such that f(x) = g(x), f(x) = constant, or f(x) = 0.

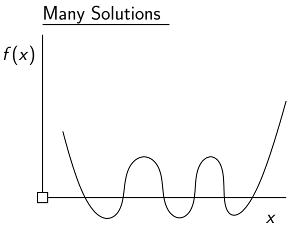


All forms may be put in the format F(x) = 0.

Numerical Solution of Equations

Mathematical Difficulties.





Quality of a Solution

Several possibilities exist:

- Solution x^* is good if $f(x^*) \approx 0.0$
- Solution x^* is good if it is close to the exact answer.

Easy to find functions that satisfy one criteria, but not both.

Numerical Solution of Equations

Example 1. Consider the equation:

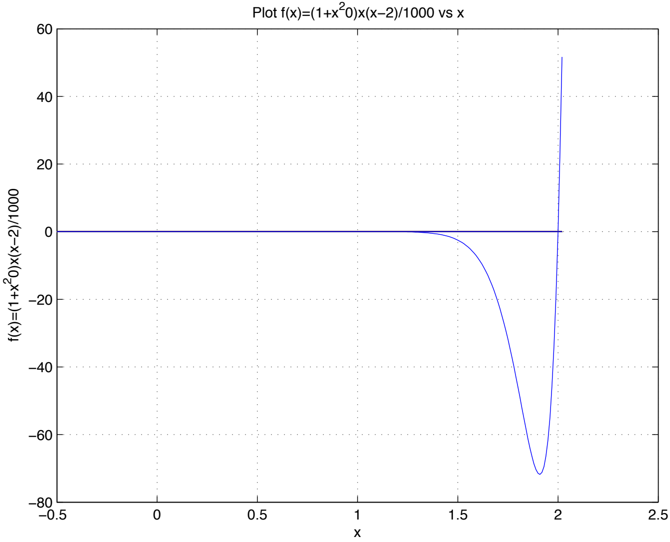
$$f(x) = \left[\frac{(x^{20}+1)x(x-2)}{1000}\right] \tag{1}$$

We know x = 0 and x = 2 are roots, but:

- x = 0.123 satisfies (i) but not (ii).
- x = 2.001 satisfies (ii) but not (i).

x	F(x)
0.123	-2.31×10^{-4}
2.001	2.1200
0.000	0.0000
2.000	0.0000

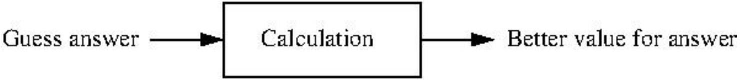
Numerical Solution of Equations



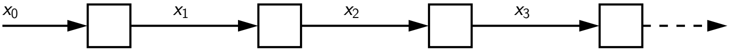
Iterative Methods

Iterative Methods

Procedure. Solve problem through a sequence of approximations:



Apply process iteratively:



Ideally, $x_0, x_1, \cdots, x_n$ will converge to the true answer.

Potential problems:

- Sequence may not converge.
- Convergence may be slow.

Iterative Methods

Example 1. Divide-and-average method for computing $\sqrt{A}$ is equivalent to solving:

$$x^2 = A \Longrightarrow x = \frac{A}{x} \Longrightarrow \frac{1}{2}\left[x + \frac{A}{x}\right] \Longrightarrow x_{n+1} = \frac{1}{2}\left[x_n + \frac{A}{x_n}\right]. \quad (2)$$

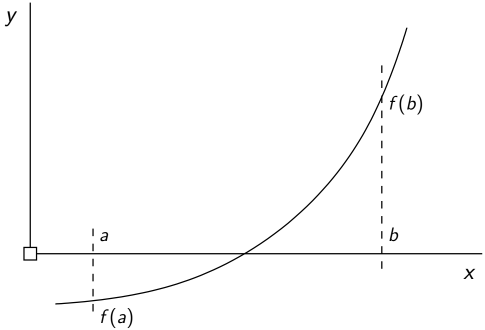
Let A $= 4$. Use initial guess $x_1 = 1 \approx \sqrt{4}$.

n	x_n	x_{n+1}
1	1.0000	2.5000
2	2.5000	2.0500
3	2.0500	2.0060
4	2.0060	2.0000

Problem Solving Strategies

Problem Solving Strategies

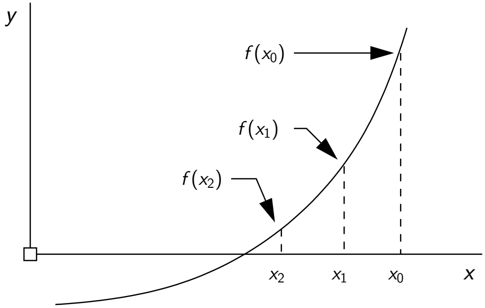
Bracketing Methods: Requires two initial guesses that bracket the solution.



- Various algorithms for computing estimates to $f(x) = 0$, e.g, Bisection, Secant stiffness.

Problem Solving Strategies

Open Methods: Methods may involve one or more initial guesses, but no need to bracket a solution.



- Algorithms are designed to provide updates: Newton Raphson Iteration, Modified Newton Raphson.