Matrices and Vectors: Basic Introduction

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ENCE 201, Fall Semester 2023

September 27, 2023

Overview

Part 1

Definition of Matrices

Definition of a Matrix

Definition. A matrix (or array) of order m by n is simply a set of numbers arranged in a rectangular block of m horizontal rows and n vertical columns. We say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (1)$$

is a matrix of size (or dimension) $(\mathtt{m} \times \mathtt{n})$.

In the double subscript notation a_{ij} for matrix element a(i,j), the first subscript i denotes the row number, and the second subscript j denotes the column number.

Matrix Properties

Matrix Properties

Properties of Matrix A:

- A matrix having the same number of rows and columns is called square.
- A square matrix of order n is also called a $(n \times n)$ matrix.
- The elements $a_{11}, a_{22}, \cdots, a_{nn}$ are called the principal diagonal.
- A diagonal matrix with elements $a_{ii} = 1$, and all other matrix elements zero, is called the identity matrix I.

Matrix Transpose

Matrix Transpose. The transpose of a $(m \times n)$ matrix A is the $(n \times m)$ matrix obtained by interchanging the rows and columns of A. The tranpose is denoted A^T.

Example 1. The matrix transpose of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \quad \text{is} \quad A^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix} \tag{2}$$

Properties

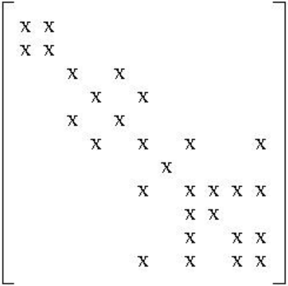
- $(A + B)^T = A^T + B^T$.
- $(ABC)^T = C^T B^T A^T$.

Symmetric and Skew-Symmetric Matrices

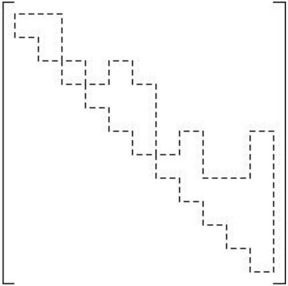
Matrix Symmetry:

- A square matrix A is symmetric if A $= A^T$.
- A square matrix A is skew-symmetric if A $=$ -A^T.

Large symmetric matrices play a central role in structural analysis.



Schematic of Non–Zero Matrix Elements



Skyline Storage Pattern

Matrix Inverse

Definition: When it exists, the inverse of matrix A is written A^{-1} and it has the property:

$$[A]\left[A^{-1}\right] = \left[A^{-1}\right][A] = I. \tag{3}$$

Nomenclature

- If matrix A has an inverse, then A is called non-singular.
- If matrix A has an inverse, then the inverse will be unique.
- If matrix A does not have an inverse, then A is called singular.

Theorem. For a $(n \times n)$ matrix A, the inverse A^{-1} exists $\Longleftrightarrow$ rank(A) = n.

- Conversely, matrix A is singular if rank(A) $<$ n (i.e., rank deficient).

Matrix Inverse

Computational Procedure. We want to carry out row operations such that:

$$[A|I] \xrightarrow{\text{row operations}} \left[I|A^{-1}\right]. \tag{4}$$

Example. Can apply row operations to get:

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{row ops}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]. \tag{5}$$

If A has rank(A) $<$ n, then the last row in echelon form will be the O (zero) vector, and the computation will fail.

Matrix Inverse

Properties:

$$\left[A^{-1}\right]^{-1} = A. \tag{6}$$

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{7}$$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}. \tag{8}$$

$$\left[A^{T}\right]^{-1} = \left[A^{-1}\right]^{T}. \tag{9}$$

Lower and Upper Triangular Matrices

A lower triangular matrix `L` is one where $a_{ij} = 0$ for all entries above the diagonal.

An upper triangular matrix `U` is one where $a_{ij} = 0$ for all entries below the diagonal. That is,

$$L = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mn} \end{bmatrix} \tag{10}$$

Matrix Arithmetic

Matrix Addition and Subtraction

Definition. If A is a $(m \times n)$ matrix and B is a $(r \times p)$ matrix, then the matrix sum $C = A + B$ is defined only when $m = r$ and $n = p$, and is a $(m \times n)$ matrix C whose elements are

$$c_{ij} = a_{ij} + b_{ij}, \text{ for } i = 1, 2, \cdots m \text{ and } j = 1, 2, \cdots n. \tag{11}$$

Properties

- (kA) B = k $(A.B)$
- $A(BC) = (AB)C.$
- $(A + B)C) = AB + AC.$
- $C(A + B) = CA + CB.$

Matrix Addition and Subtraction

Example 1. Let

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix}. \tag{12}$$

The matrix sum is:

$$C = A + B = \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 4 & 7 \end{bmatrix}. \tag{13}$$

Matrix Multiplication

Definition. Let A and B be $(m \times n)$ and $(r \times p)$ matrices, respectively.

The matrix product $A \cdot B$ is defined only when interior matrix dimensions are the same (i.e., $n = r$).

The matrix product C = A · B is a $(m \times p)$ matrix whose elements are

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \ b_{kj} \tag{14}$$

for $i = 1, 2, \cdots m$ and $j = 1, 2, \cdots n$.

Matrix Multiplication

Example 1. Assuming that matrices A and B are as defined in the previous section:

$$
\begin{aligned}
\mathtt{C} = \mathtt{A} \cdot \mathtt{B} &= \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \cdot 4 + 1 \cdot 0 & 2 \cdot 2 + 1 \cdot 1 \\ 4 \cdot 4 + 6 \cdot 0 & 4 \cdot 2 + 6 \cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} 8 & 5 \\ 16 & 14 \end{bmatrix}.
\end{aligned} \tag{15}
$$

Geometric Interpretation. Matrix element c_{ij} is the dot product of the i-th row of A with the j-th column of B.

Matrix Multiplication

Properties.

- A.B.C = (A.B).C = A.(B.C).
- A.(B + C) = A.B + A.C.
- (A + B).C = A.C + B.C.
- A.I = A.
- In general, A.B $\neq$ B.A.
- A.B = ϕ does not necessarily imply A = ϕ or B = ϕ. Counter example:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{16}$$