Linear Matrix Equations – Part 1

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Overview

Part 3

Linear

Matrix Equations

Linear Matrix Equations

Matrix Form. The matrix counterpart of 1 is $[A] \cdot [X] = [B]$, where

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \\ \vdots \\ b_m \end{bmatrix} \quad (2)$$

Points to note:

- Matrices A and X have dimensions $(m \times n)$ and $(n \times 1)$, respectively.
- Column vector B has dimensions $(n \times 1)$.

Analysis of Solutions to Matrix Equations

Key Observations

- For two- and three-dimensions, graphical methods and intuition work well.
- For problems beyond three dimensions, much more difficult to understand the nature of solutions to linear matrix equations.
- We need to rely on mathematical analysis instead.

Basic Questions

- How many solutions will a set of equations will have?
- How to determine when no solutions exist?
- If there is more than one solution, how many solutions exist?

Fortunately, hand calculations on very small systems can provide hints on a pathway forward.

Elementary Row Operations

Elementary Row Operations

Purpose. An elementary row operation transforms the structure of matrix equations $[A][X] = [B]$ without affecting the underlying solution $[X]$.

Three Types of Elementary Row Operation

- Swap any two rows.
- Multiply any row by a non-zero number.
- Add to one equation a non-zero multiple of another equation.

Are they Useful?

- Yes! Elementary row operations are used in Gaussian Elimination to reduce a matrix $[A]$ to row echelon form (much easier to work with).

Elementary Row Operations

Example 1. Swap rows 1 and 3:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}. \tag{22}$$

Example 2. Replace row 2 by itself minus 2 times row 1:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow{r_2 \to r_2 - 2r_2} \begin{bmatrix} a & b & c \\ d - 2a & e - 2b & f - 2c \\ g & h & i \end{bmatrix}. \tag{23}$$

Elementary Row Operations

Row Operations Modeled as Matrix Transformations. Each of these operations can be viewed in terms of an elementary matrix transformation $[E]$, e.g.,

$$A_0 \xrightarrow{\text{row operation}} A_1 \Longleftrightarrow EA_0 \rightarrow A_1. \tag{24}$$

We can design a sequence of transformation matrices E_1, E_2 ... E_n, i.e.,

$$[E_n] \cdots [E_2] [E_1] [A] [X] = [E_n] \cdots [E_2] [E_1] [B], \tag{25}$$

to simplify (upper triangular form) the matrix structure of the left-hand side.

Elementary Row Operations

Example 1. This transformation that swaps rows 1 and 3.

$$[E]\,[A] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}. \qquad (26)$$

Example 2. The matrix transformation:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \begin{bmatrix} a+2d & b+2e & c+2f \\ d & e & f \\ g & h & i \end{bmatrix}. \qquad (27)$$

replaces row 1 by itself + two times row 2.

Elementary Row Operations

Example 3. Starting from the augmented matrix $[A|I]$, we can design sequences of elementary row operations to compute $\left[I|A^{-1}\right]$, i.e.,

$$[A|I] \xrightarrow{\text{row operations}} \left[I|A^{-1}\right]. \tag{28}$$

Here is a simple example:

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{row ops}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]. \tag{29}$$

Elementary Row Operations

Example 4. The computational procedure

$$[A|I] \xrightarrow{\text{row operations}} \left[I|A^{-1}\right]. \tag{30}$$

fails for matrix equations that are either inconsistent or overlapping.

Consider the pair of equations: $x_1 + x_2 = 2.0$ and $x_1 + x_2 = 1.0$. Applying row operations to the augmented form gives:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right]. \tag{31}$$

We conclude that $\left[A^{-1}\right]$ does not exist. The equations are either inconsistent or overlapping.

Echelon Form

Echelon Form

Definition

- The first no-zero entry of a row (or column) is called the leading entry.

Definition of Echelon Form

A matrix is in echelon form (i.e., upper triangular form) if:

- All non-zero rows are above any zero row (i.e., a row with all zeros).
- For any two rows, the column containing the leading entry of the upper row is on the left of the column containing the leading entry of the lower row.

Matrices in echelon form display an upper triangular pattern.

Echelon Form

Example 1. Two matrices in echelon form:

$$\begin{bmatrix} 1 & 4 & 0 & 6 & 10 & 6 \\ 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 0 & 6 & 10 & 6 \\ 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (32)$$

Example 2. These matrices are not in echelon form:

$$\begin{bmatrix} 1 & 4 & 0 & 6 & 10 & 6 \\ 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 3 & 1 & 2 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 1 & 0 & 2 & 1 \\ 1 & 4 & 0 & 6 & 10 & 6 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}. \qquad (33)$$

Reduced Echelon Form

Definition of Reduced Echelon Form

A matrix is in reduced echelon form if in addition to the criteria stated above:

- All leading entries are 1, and they are the only non-zero entries in each pivot (i.e., leading entry) column.

Any matrix can be reduced by a sequence of elementary row operations to a unique reduced Echelon form.

Example 1. Matrices in reduced Echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 2 & 0 & 10 & 6 \\ 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

Matrix Rank

Matrix Rank

Definition. The rank of a matrix A, denoted rank(A), is the dimension of the vector space spanned by its rows (or columns).

- It is the number of linearly independent rows (or columns) in the matrix.

Theorem. For a $(n \times n)$ matrix A, the inverse A^{-1} exists $\Longleftrightarrow$ rank(A) $=$ n. Conversely, matrix A is singular if rank(A) $<$ n (i.e., rank deficient).

Computational Procedure. The standard way of determining the rank of a matrix is to:

- Transform the matrix to row echelon form.
- The rank is equal to the number of rows containing non-zero elements.

Matrix Rank

Example 1. The matrix

$$A = \begin{bmatrix} 3 & 1 & 9 \\ 1 & -2 & 5 \\ 2 & 3 & 4 \end{bmatrix}, \tag{35}$$

Applying row operations gives:

$$\begin{bmatrix} 3 & 1 & 9 \\ 1 & -2 & 5 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2 - R_3} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 5 \\ 2 & 3 & 4 \end{bmatrix}. \tag{36}$$

A has rank 2 because (by construction) row 1 is the sum of rows 2 and 3 (i.e., $row_1 - row_2 - row_3 = 0$).

Matrix Rank

Example 2. Let $x_1 + x_2 = 2.0$ and $x_1 + x_2 = 1.0$. Applying row operations to $[A|B]$ gives:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \rightarrow R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & -1 \end{array}\right]. \qquad (37)$$

Inconsistent: $[A]$ is singular and rank $[A|B] \neq$ rank $[A]$.

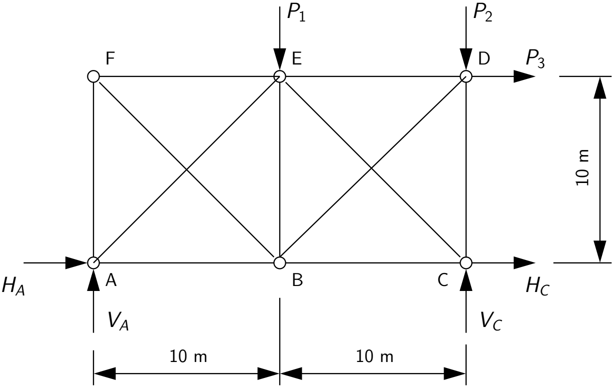
Example 3. Let $x_1 + x_2 = 2.0$ and $2x_1 + 2x_2 = 4.0$. Applying row operations to $[A|B]$ gives:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 2 & 4 \end{array}\right] \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]. \qquad (38)$$

Overlapping: $[A]$ is singular and rank $[A|B]$ equals rank $[A]$.

Matrix Rank

Example 4. Consider equilibrium of the pin-jointed frame subject to external loads P_1, P_2 and P_3.



We wish to know the reactions as a function of applied forces.

Matrix Rank

Equations of equilibrium (not completely correct):

$$\sum H = 0 \to H_A + H_C + P_3 = 0.$$
$$\sum V = 0 \to V_A + V_C - P_1 - P_2 = 0.$$
$$\sum M_A = 0 \to -20V_C + 10P_1 + 20P_2 + 10P_3 = 0.$$
$$\sum M_C = 0 \to 20V_A - 10P_1 + 10P_3 = 0.$$

Writing the equations in matrix form:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -20 & 0 \\ 20 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_A \\ H_A \\ V_C \\ H_C \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 10 & 20 & 10 \\ -10 & 0 & 10 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{39}$$

Matrix Rank

Symbolically, we have a set of matrix equations:

$$[A]\,[R] + [B]\,[P] = [0]\,. \tag{40}$$

If $\left[A^{-1}\right]$ exists, then we have:

$$[R] = -\left[A^{-1}\right][B]\,[P]\,. \tag{41}$$

Apply the following sequence of row operations:

- Scale row 3: $R_3 \to -R_3/20$
- Scale row 4: $R_4 \to R_4/20$
- Subtract rows 3 and 4 from row 2: $R_2 \to R_2 - R_3 - R_4$.
- Swap rows: $R_2 \longleftrightarrow R_4$, then $R_2 \longleftrightarrow R_1$.

Matrix Rank

Summary of Row Operations:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -20 & 0 \\ 20 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{42}$$

Conclusion:

- Because matrix A is rank deficient (i.e. rank(A) $= 3 < 4$), the matrix inverse $\left[A^{-1}\right]$ does not exist, and a unique solution to this problem cannot be found.
- The error lies in the use of $\sum M_A = 0$ and $\sum M_C = 0$ – they are not independent.