Linear Matrix Equations – Part 1

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ENCE 201, Fall Semester 2023

October 6, 2023

Overview

Part 1

Linear

Matrix Equations

Linear Matrix Equations

Definition. A system of m linear equations with n unknowns may be written

$$\begin{array}{cccccccccc} a_{11}x_1 & + a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & & & \vdots & & & \ldots & & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \tag{1}$$

Points to note:

- The constants $a_{11}, a_{21}, a_{31}, \cdots a_{mn}$ and $b_1, b_2, \cdots b_m$ are called the equation coefficients.
- The variables $x_1, x_2 \cdots x_n$ are the unknowns in the system of equations.

Linear Matrix Equations

Matrix Form. The matrix counterpart of 1 is $[A] \cdot [X] = [B]$, where

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad (2)$$

Points to note:

- Matrices A and X have dimensions $(m \times n)$ and $(n \times 1)$, respectively.
- Column vector B has dimensions $(n \times 1)$.

Augmented Matrix Form

Augmented Matrix Form. An augmented matrix for a system of equations is matrix A juxtiposed with matrix B.

Example. The augmented matrix form form of equation 2 is:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & & \vdots & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} & b_m \end{array}\right] \tag{3}$$

The augmented matrix dimensions are $(m \times (n+1))$.

Definition of Linear

Mathematical Definition. Let k be a non-zero constant. A function y = f(x) is said to be linear if it satisfies two properties:

- $y = f(kx_1)$ is equal to $y = kf(x_1)$.
- $f(x_1 + x_2) = f(x_1) + f(x_2)$.

For constants k and m these equations can be combined:

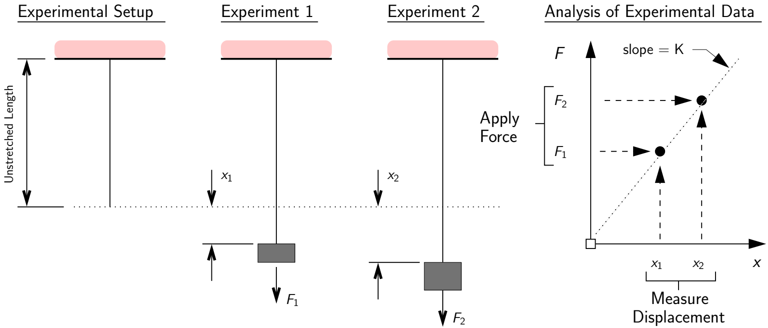
$$kf(x_1) + mf(x_2) \to f(kx_1 + mx_2). \tag{4}$$

Economic Benefit. Often evaluation of $y = f(x)$ has a cost.

Linearity allows us to compute $y_1 = f(x_1)$ and $y_2 = f(x_2)$ and then predict the system response for $kx_1 + mx_2$ via linear combination of solutions. This is free!

Definition of Linear

Example 1. Consider an experiment to determine the extension of an elastic chord as a function of applied force.



Linearity allows us to predict solutions:

$$Kx_1 = F_1, Kx_2 = F_2, \rightarrow K(mx_1 + nx_2) = mF_1 + nF_2. \quad (5)$$

Definition of Linear

Example 2. Analysis of Linear Structural Systems (ENCE 353):

Let matrix equations $AX = B$ represent behavior of a structural system:

- Matrix A will capture the geometry, material properties, etc.
- Matrix B represents externally applied loads (e.g., dead/live gravity loads).
- Column vector X represents nodal displacements.

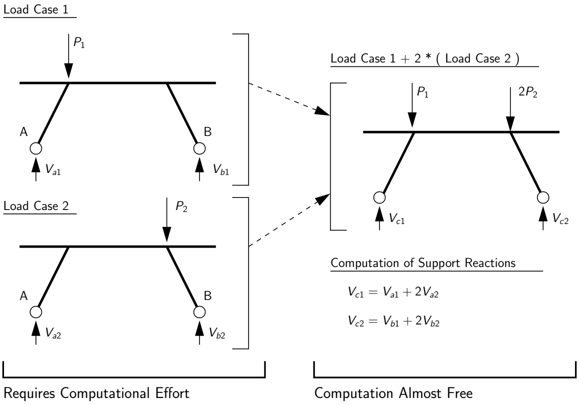
Solving $AX = B$ requires computational work $O(n^3)$.

However, if matrix system is linear, then:

$$AX_1 = B_1, AX_2 = B_2 \rightarrow A(mX_1 + kX_2) = mB_1 + kB_2. \quad (6)$$

Definition of Linear

We can simply add the results of multiple load cases:



Works for support reactions, bending moments, displacements, etc.

Solutions in Two and Three Dimensions

Equations in Two Dimensions

Let $m = n = 2$.

The pair of equations:

$$a_{11}\,x_1 + a_{12}\,x_2 = b_1 \tag{7}$$
$$a_{21}\,x_1 + a_{22}\,x_2 = b_2 \tag{8}$$

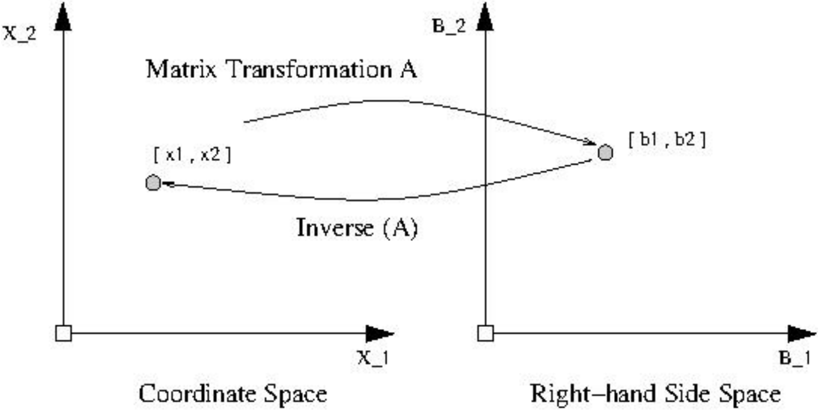
can be interpreted as a pair of straight lines in the (x_1, x_2) plane.

The equations in matrix form are:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{9}$$

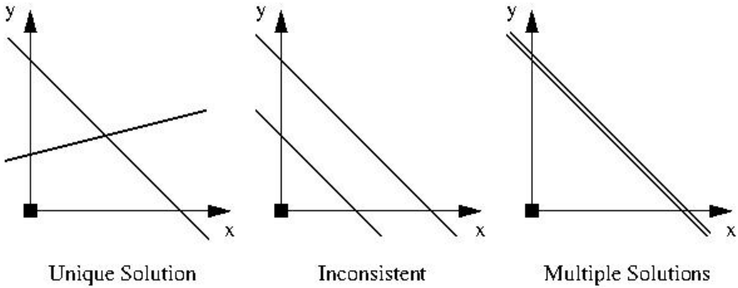
Equations in Two Dimensions

Matrix Tranformation: $[A]\,[X] \rightarrow [B]$.



Equations in Two Dimensions

Three Types of Solutions:



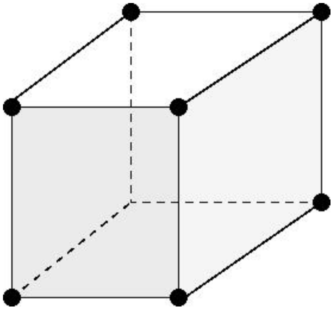
- Unique solution when two lines meet at a point.
- No solutions when two lines are parallel but not overlapping.
- Multiple solutions when two lines are parallel and overlap.

Equations in Three Dimensions

Also Three Types of Solutions:

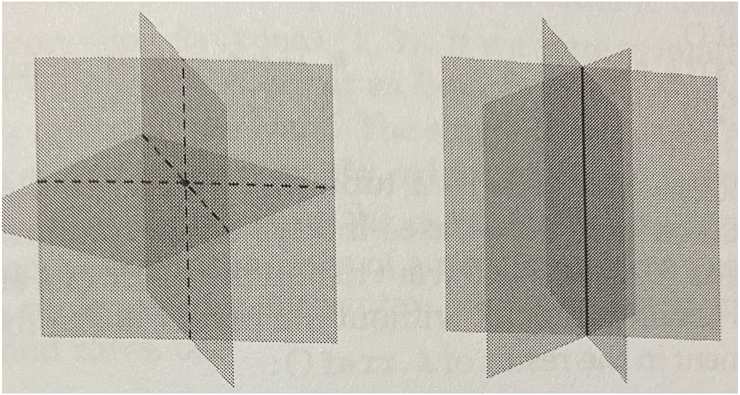
Each equation corresponds to a plane in three dimensions (e.g., think walls, floor and ceiling in a room).

- Unique solution when three planes intersect at a corner point.
- Multiple solutions where three planes overlap or meet along a common line.
- No solutions when three planes are parallel, but distinct, or pairs of planes that intersect along a line (or lines).



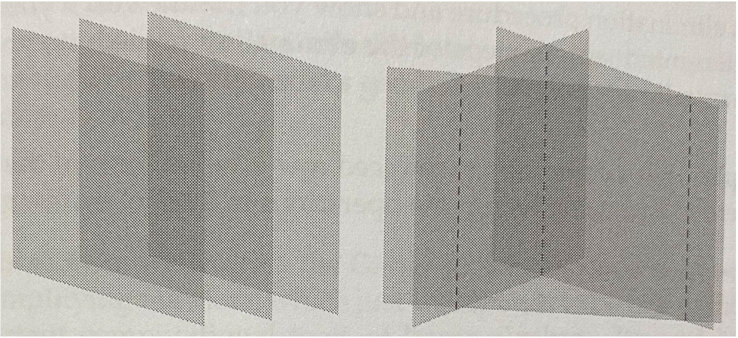
Equations in Three Dimensions

One Solution/Infinite Solutions:



Equations in Three Dimensions

No Solutions:



Analysis of Solutions to Matrix Equations

Key Observations

- For two- and three-dimensions, graphical methods and intuition work well.
- For problems beyond three dimensions, much more difficult to understand the nature of solutions to linear matrix equations.
- We need to rely on mathematical analysis instead.

Basic Questions

- How many solutions will a set of equations will have?
- How to determine when no solutions exist?
- If there is more than one solution, how many solutions exist?

Fortunately, hand calculations on very small systems can provide hints on a pathway forward.