Department of Civil and Environmental Engineering, Fall Semester, 2023

ENCE 201 Midterm 1, Open Notes and Open Book

Name : Austin

E-mail (print neatly!): austin@umd.edu.

Exam Format and Grading. This take home midterm exam is open notes and open book. You need to comply with the university regulations for academic integrity.

There are three questions. Partial credit will be given for partially correct answers, so please show all your working.

Please see the **class web page for instructions on how to submit your exam paper**.

Question	Points	Score
1	13	
2	15	
3	12	
Total	40	

Question 1: 13 points. This question covers linear matrix equations and simple matrix operations. Let's begin by considering the family of m linear equations with n unknowns:

$$\begin{array}{llllll} a_{11}x_1 & + a_{12}x_2 & + a_{13}x_3 + & \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 & + a_{22}x_2 & + a_{23}x_3 + & \cdots + a_{2n}x_n & = b_2 \\ a_{31}x_1 & + a_{32}x_2 & + a_{33}x_3 + & \cdots + a_{3n}x_n & = b_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + a_{m3}x_3 + & \cdots + a_{mn}x_n & = b_m \end{array} \tag{1}$$

[1a] (3 pts) Suppose that $m = n = 3$. Write equations 1 in matrix form A.X = B.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

[1b] (2 pts) In part [1a], what are the dimensions of matrices A, X and B?

A X = B

(3x3)(3x1) (3x1)

[1c] (3 pts) Write down the augmented matrix $[A|B]$.

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

[1d] (3 pts) Now suppose that matrix A is symmetric. What constraints does symmetry place on the number of rows and columns in matrix A, and the values of the matrix elements.

Symmetry implies $a_{ij} = a_{ji}$.

$$\Rightarrow \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A.$$

[1e] (2 pts) Consider the matrices

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad and \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}. \tag{2}$$

Why is the matrix product $B \cdot C$ undefined?

$$B.C = \underset{(2 \times 3)}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{(2 \times 2)}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$$

Inside dimensions need to have same value in order for matrix multiply to work.

Question 2 (15 points). This question covers use of Python to compute the polar moment of inertia for a small collection of masses.

Figure 1 shows the spatial layout and approximate size of the massses.

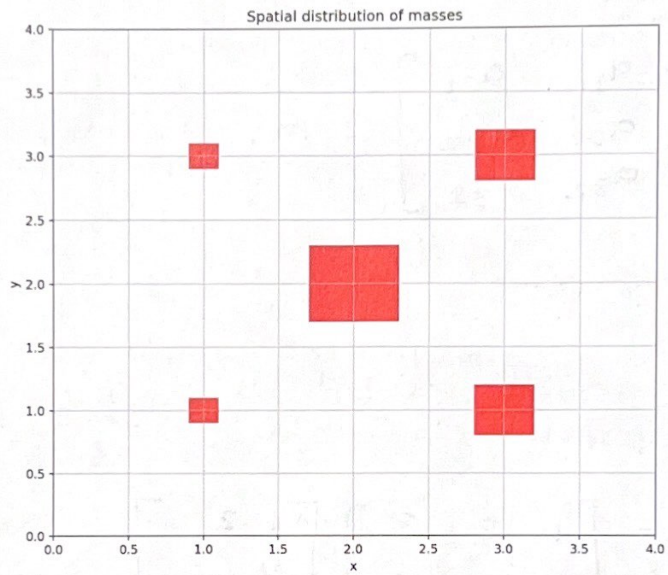

Figure 1. Two-dimensional grid of masses.

If the total number of point masses is denoted by N, then the polar moment of inertia (as measured about the origin) is given by

$$I_{rr} = \sum_{i=1}^{N} \left[x_i^2 + y_i^2 \right] \cdot m_i \tag{3}$$

The following Python program evaluates equation 3 and generates Figure 1. Look the code over carefully and answer the questions that follow:

```
# ==============================================================
# TestMomentsInertia.py: Compute Irr.
#
# Written by: Mark Austin                          October 2023
# ==============================================================

import numpy as np
import matplotlib.pyplot as plt
from matplotlib.patches import Rectangle
```

```
# Main function ...

def main():

    # Mass and coordinate arrays

    mass  = np.array( [ 1.0, 2.0, 3.0, 2.0, 1.0 ]  )
    coord = np.array( [ ( 1.0, 1.0 ),
                        ( 3.0, 3.0 ),
                        ( 2.0, 2.0 ),
                        ( 3.0, 1.0 ),
                        ( 1.0, 3.0 ) ] );

    # Simple approach to computing polar moment of inertia ...

    Irr = 0.0;
    for i in range(len(mass)):
        xcoord = coord[i][0];
        ycoord = coord[i][1];
        Irr = Irr + mass[i]*(xcoord*xcoord + ycoord*ycoord)

    # Print results ...

    print("--- Irr = {:10.2f}  ...".format(Irr) );

    # Plot masses and coordinates ...
    # Define Matplotlib figure and axis

    fig, ax = plt.subplots()

    # Draw masses as small/medium-sized rectangles ...

    for i in range(len(mass)):

        xcoord = coord[i][0]; ycoord = coord[i][1];

        dm = mass[i];
        if dm == 1:
           width = 0.2;
        elif dm == 2:
           width = 0.4;
        else:
           width = 0.6;

        ax.add_patch(Rectangle( (xcoord - width/2, ycoord-width/2), width, width, facecolor='red')

    plt.title('Spatial distribution of masses')
    plt.ylabel('y')
    plt.xlabel('x')
    plt.ylim( 0, 4)
    plt.xlim( 0, 4)
    plt.grid(True)
    plt.show()
```

```
# call the main method ...

main()
```

[2a] (3 pts) What does the line:

```
import numpy as np
```

do, and why is it needed in this program?

Imports numpy (a library that performs common mathematical operations such array functions & linear algebra) as np, Needed to store arrays mass and coord.

[2b] (3 pts) Write a small script of Python code to retrieve and print the matrix dimensions, and number of rows and columns in matrices mass and coord.

```
print("mass; no dimen = {:d}".format(mass.ndim))
  "      "  : no rows  = {:d}".format(mass.shape[0]))
  "    (" coord : no dimen = {:d}".format(coord.ndim))
                       :      coord.shape[0]))
                       :      coord.shape[1])).
```

[2c] (3 pts) Now let's consider the block of code:

```
Irr = 0.0;
for i in range(len(mass)):
    xcoord = coord[i][0];
    ycoord = coord[i][1];
    Irr = Irr + mass[i]*(xcoord*xcoord + ycoord*ycoord)
```

Create a table that shows the iteration value i, and values of xcoord, ycoord, and Irr, for each iteration of the loop computation.

i	xcoord	ycoord	mass[i]	Irr
0	1	1	1	2
1	3	3	2	38
2	2	2	3	62
3	3	1	2	82
4	1	3	1	92.

[2d] (3 pts) Briefly explain how the formatting specification in:

```
print("--- Irr = {:10.2f}  ...".format(Irr) );
```

works. 'What will the output look like? one-to-one match.

{:10.2f} — print as a floating-point number, 10 chars wide, with 2 decimal places of accuracy after decimal point

Irr = 92.00

10 characters.

[2e] (3 pts) Finally, consider the block of code:

```
for i in range(len(mass)):

    xcoord = coord[i][0]; ycoord = coord[i][1];

    dm = mass[i];
    if dm == 1:
       width = 0.2;
    elif dm == 2:
       width = 0.4;
    else:
       width = 0.6;

    ax.add_patch(Rectangle( (xcoord - width/2, ycoord-width/2), width, width, facecolor='
```

Briefly explain how the rectangle width and lower left-hand coordinate points are computed for each of the masses. A table format would be good.

i	dm	x	y	width	$x - w/2$	$y - w/2$
0	1	1	1	0.2	0.9	0.9
1	2	3	3	0.4	2.8	2.8
2	3	2	2	0.6	1.7	1.7
3	2	3	1	0.4	2.8	0.8
4	1	1	3	0.2	0.9	2.9.

Question 3 (12 points). This question ccvers linear algebra and its application to the solution of linear matrix equations.

[3a] (3 pts) Find all of the solutions to:

$$\begin{bmatrix} x+y & 2 \\ 1 & 4x-2y \end{bmatrix} = \begin{bmatrix} 2 & x+y \\ x-y & 5 \end{bmatrix}. \tag{4}$$

Both sides of (4) need to be equal.

$\Rightarrow \quad x+y = 2 \quad — \quad A$

$x-y = 1 \quad — \quad B$

$4x-2y = 5 \quad — \quad C.$

$A + B. \quad 2x = 3 \Rightarrow x = 3/2, \; y = \frac{1}{2}$

Check with C. $\quad 4x-2y = \frac{12}{2} - \frac{2}{2} = 5$ ✓

[3b] (3 pts) If $x = [1, 2, 3]^T$ (i.e., it's matrix with only one column, also called a column vector), evaluate $x^T \cdot x$ and $x \cdot x^T$.

$$\underset{(1\times3)(3\times1)}{x^T . x} = (1, 2, 3)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 1+4+9 = 14.$$

$$\underset{(3\times1)(1\times3)}{x . x^T} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}[1, 2, 3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

[3c] (3 pts) If $[A] = [I] - y \cdot y^T$, where y is a column vector, calculate $[A]^2$ in terms of y and y^T. Hence, deduce that if $y^T \cdot y = 2$ then $[A]^2 = [I]$. Note: This question is tricky – show all of your working, and no cheating !!!

$$A = I - y \cdot y^T$$
$$(n \times n) \quad (n \times n)(n \times 1)(1 \times n).$$

$$A^2 = (I - y \cdot y^T)(I - y \cdot y^T)$$
$$= I^2 - I \cdot y \cdot y^T - y \cdot y^T \cdot I + y \cdot \underbrace{y^T \cdot y}_{2} \cdot y^T$$
$$= I - 2yy^T + 2yy^T$$
$$= I.$$

[3d] (3 pts) Show that for all values of x, y and $a \geq 0$ the triple product of matrices:

$$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 1+a & -a \\ -a & a \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \geq 0 \qquad (5)$$

$(1 \times 2)(2 \times 2)(2 \times 1) \longrightarrow$ evaluates to a number!

$$(x, y)\begin{bmatrix} 1+a & -a \\ -a & a \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (1+a)x - ay, & -ax + ay \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow (1+a)x^2 - axy - axy + ay^2$$
$$\Rightarrow x^2 + a(x^2 - 2xy + y^2)$$
$$\Rightarrow x^2 + a(x-y)^2 \geq 0 \checkmark.$$