On Mixed-μ Synthesis

André L. Tits
Department of Electrical and Computer Engineering
and Institute for Systems Research
University of Maryland, College Park
MD 20742, USA

Yung-Shan Chou
Department of Electrical Engineering
Tamkang University
Tamsui, Taipei Hsien, Taiwan 25137

August 23, 1999

Abstract

An identity invoked implicitly in P.M. Young's (D,G)-K iteration for mixed-μ synthesis is formally stated and a direct, conceptual proof is provided. The identity holds both in discrete-time and in continuous-time. Connection with the (D,G)-K iteration is explicited and variations on the iteration are suggested.

1 Introduction and Preliminaries

An identity invoked implicitly in P.M. Young's (D,G)-K iteration for mixed-μ synthesis [1] (see also Section 18.2 in [2]) is formally stated and a direct, conceptual proof is provided. The identity holds both in discrete-time and in continuous-time. Connection with the (D,G)-K iteration is explicited and variations on the iteration are suggested. In the special case of complex-μ

synthesis the identity can be found, e.g., in [3, 2], but to our knowledge even in that case no proof is available to date in the open literature.

We focus on the discrete-time case. $\mathbf{D}$ denotes the open unit disk $\{z \in \mathbf{C} : |z| < 1\}$, $\partial\mathbf{D}$ the unit circle, and $\mathbf{D}^{\mathrm{c}}$ the complement of $\mathbf{D}$. $\mathbf{RL}_\infty$ denotes the set of proper real-rational transfer matrices with no poles on $\partial\mathbf{D}$ and $\mathbf{RH}_\infty$ the set of proper real-rational transfer matrices with no poles in $\mathbf{D}^{\mathrm{c}}$. Given a complex matrix M, M^* denotes its complex conjugate transpose and, given a real-rational transfer matrix P, its adjoint $P^\sim$ is defined by $P^\sim(z) = P(1/z)^T$, where subscript T denotes transposition.

Given positive integers k_{r}, k_{c}, k_{C} and n, with $k_{\mathrm{r}} + k_{\mathrm{c}} + k_{\mathrm{C}} = n$, consider the matrix subspaces

$$\tilde{\mathcal{D}} := \{\mathrm{diag}(D^{\mathrm{r}}, D^{\mathrm{c}}, d^{\mathrm{C}} I_{k_{\mathrm{C}}}) : D^{\mathrm{r}} \in \mathbf{C}^{k_{\mathrm{r}} \times k_{\mathrm{r}}}, D^{\mathrm{c}} \in \mathbf{C}^{k_{\mathrm{c}} \times k_{\mathrm{c}}}, d^{\mathrm{C}} \in \mathbf{C}\},$$

$$\tilde{\mathcal{G}} := \{\mathrm{diag}(G^{\mathrm{r}}, 0_{k_{\mathrm{c}}+k_{\mathrm{C}}}) : G^{\mathrm{r}} \in \mathbf{C}^{k_{\mathrm{r}} \times k_{\mathrm{r}}}\}.$$

In the structured singular value framework the above correspond to the case where the uncertainty structure consists of a single block of each type (repeated scalar real, repeated scalar complex, full matrix complex) of respective sizes k_{r}, k_{c} and k_{C}. All of the arguments and results presented in this note apply without modifications to the general case, where each nonzero block in $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{G}}$ is restricted to be a block-diagonal matrix with blocks of prescribed sizes equal to those of the respective uncertainty blocks. See, e.g., Section 18.2 of [2] for details. Subspaces $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{G}}$ are defined here for the simpler case for sake of economy of notation.

The following subsets of $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{G}}$ are of special interest:

$$\mathcal{D}^+ := \{D \in \tilde{\mathcal{D}} : D^* = D, D > 0\},$$

$$\mathcal{G} := \{G \in \tilde{\mathcal{G}} : G^* = -G\}.$$

Given a matrix $M \in \mathbf{C}^{n \times n}$, the mixed-$\mu$ upper bound $\hat{\mu}(M)$ of M associated with the uncertainty structure under consideration is defined by

$$\hat{\mu}(M) := \inf_{\substack{\beta \geq 0 \\ D \in \mathcal{D}^+ \\ G \in \mathcal{G}}} \left\{ \beta : M^* D M + G M - M^* G - \beta^2 D < 0 \right\}.$$

Finally, we will make extensive use of the following subsets of $\mathbf{RH}_\infty$ and $\mathbf{RL}_\infty$ associated with the uncertainty structure under consideration:[1]

$$\mathcal{D}^{\mathrm{U}}_{\mathrm{H}_\infty} := \{D \in \mathbf{RH}_\infty : D^{-1} \in \mathbf{RH}_\infty, D(z) \in \tilde{\mathcal{D}}\ \forall z \in \mathbf{D}^{\mathrm{c}}\},$$

$$\mathcal{D}^{+}_{\mathrm{L}_\infty} := \{D \in \mathbf{RL}_\infty : D(z) \in \mathcal{D}^{+}\ \forall z \in \partial\mathbf{D}\},$$
$$\mathcal{G}_{\mathrm{L}_\infty} := \{G \in \mathbf{RL}_\infty : G(z) \in \mathcal{G}\ \forall z \in \partial\mathbf{D}\}.$$

Two lemmas will be used. The first one is a restatement of implication (1)⇒(2) of Theorem 3.2 in [4].[2] The second one follows from straightforward algebra: see, e.g., Theorem 18.4 in [2].

Lemma 1 *Let* $P \in \mathbf{RH}_\infty$. *If* $\sup_{z\in\partial\mathbf{D}} \hat{\mu}(P(z)) < 1$, *then there exist* $D \in \mathcal{D}^{+}_{\mathrm{L}_\infty}$, $G \in \mathcal{G}_{\mathrm{L}_\infty}$ *such that*

$$P(z)^* D(z) P(z) + G(z)P(z) - P(z)^* G(z) - D(z) < 0 \quad \forall z \in \partial\mathbf{D}.$$

Lemma 2 *Let* $\beta > 0$, *let* M *be a complex square matrix and let* R, $\hat{R}$, S, $\hat{S}$ *be complex square matrices such that* $\hat{R} = \hat{R}^* > 0$, $\hat{S} = -\hat{S}^*$, $R^*R = \hat{R}$, *and* $\beta R^* S R = \hat{S}$. *Finally, let* T *be any matrix satisfying* $TT^* = (I + S^*S)^{-1}$. *Then*

$$M^*\hat{R}M + \hat{S}M - M^*\hat{S} - \beta^2\hat{R} < 0$$

if and only if

$$\overline{\sigma}\left((\beta^{-1}RMR^{-1} - S)T\right) < 1.$$

As per Section 7.3 of [6], given a real-rational transfer matrix P and a right coprime factorization $P = AB^{-1}$, with $A, B \in \mathbf{RH}_\infty$, (A, B) is *normalized* if

$$A^\sim A + B^\sim B = I.$$

[1] The superscript U in $\mathcal{D}^{\mathrm{U}}_{\mathrm{H}_\infty}$ stresses that the latter is a subset of the units in $\mathbf{RH}_\infty$.
[2] For the case of complex, non-repeated uncertainty, the result first appeared in [5].

Normalized right coprime factorizations always exist and are unique up to right multiplication of both factors by a same constant unitary matrix. In the sequel, we denote by $(\mathrm{ncfn}(P), \mathrm{ncfd}(P))$ an arbitrary normalized right coprime factorization of P, but make use of this notation only when the corresponding statement holds true for any such factorization.

It is readily checked (e.g, p. 388 in [2]) that given any real rational transfer matrix P,

$$(I + P^{\sim}P)^{-1} = \mathrm{ncfd}(P)\mathrm{ncfd}(P)^{\sim}. \tag{1}$$

2 μ-Synthesis Identity

The following identity can be viewed as the basis for the (D,G)-K iteration, in that it implies that minimizing its right-hand side (over stabilizing controllers) amounts to minimizing the supremum over frequency of the upper bound $\hat{\mu}(P(z))$ to the structured singular value. We adopt the normalized coprime factorization viewpoint used in [2].

Theorem 1 *Given any $P \in \mathbf{RH}_\infty$,*

$$\sup_{z\in\partial\mathbf{D}} \hat{\mu}(P(z)) = \inf_{\substack{\beta>0 \\ D\in\mathcal{D}^{\mathrm{U}}_{\mathrm{H}_\infty} \\ G\in\mathcal{G}_{\mathrm{L}_\infty}}} \left\{ \beta : \left\| [\beta^{-1}DPD^{-1} \quad - I] \begin{bmatrix} \mathrm{ncfd}(G) \\ \mathrm{ncfn}(G) \end{bmatrix} \right\|_\infty \leq 1 \right\}. \tag{2}$$

Proof: First let $\alpha > 0$ be a strict upper bound to the right-hand side of (2). Thus there exists $\beta \in (0, \alpha)$ such that, for some $D \in \mathcal{D}^{\mathrm{U}}_{\mathrm{H}_\infty}$, $G \in \mathcal{G}_{\mathrm{L}_\infty}$,

$$\overline{\sigma}\left(\left(\beta^{-1}D(z)P(z)D(z)^{-1} - G(z)\right)\mathrm{ncfd}(G)(z)\right) < 1 \quad \forall z \in \partial\mathbf{D}.$$

In view of (1), it follows from Lemma 2 that, given any $z \in \partial\mathbf{D}$, there exist $\hat{D} \in \mathcal{D}^+$, $\hat{G} \in \mathcal{G}$ such that

$$P(z)^*\hat{D}P(z) + \beta(\hat{G}P(z) - P(z)^*\hat{G}) - \beta^2\hat{D} < 0.$$

Noting that $\beta\hat{G} \in \mathcal{G}$, we conclude that

$$\sup_{z \in \partial \mathbf{D}} \hat{\mu}(P(z)) \leq \beta < \alpha.$$

Thus the left-hand side of (2) is no larger than its right-hand side. To complete the proof, let $\alpha > 0$ be such that $\sup_{z \in \partial \mathbf{D}} \hat{\mu}(P(z)) < \alpha$, i.e.,

$$\sup_{z \in \partial \mathbf{D}} \hat{\mu}(\alpha^{-1} P(z)) < 1.$$

It follows from Lemma 1 that, for some $\hat{D} \in \mathcal{D}^{+}_{\mathrm{L}_\infty}$ and $\hat{G} \in \mathcal{G}_{\mathrm{L}_\infty}$,

$$P(z)^* \hat{D}(z) P(z) + \hat{G}(z) P(z) - P(z)^* \hat{G}(z) - \alpha^2 \hat{D}(z) < 0, \ \forall z \in \partial \mathbf{D}.$$

Let D be any stable minimum phase spectral factor of $\hat{D}$, i.e., $D \in \mathcal{D}^{\mathrm{U}}_{\mathrm{H}_\infty}$ and $\hat{D} = D^{\sim} D$ (see, e.g., [7]). Thus, $\hat{D}(z) = D(z)^* D(z)$ for all $z \in \partial \mathbf{D}$ and it follows from Lemma 2 that

$$\overline{\sigma}\left(\left(\alpha^{-1} D(z) P(z) (D(z))^{-1} - G(z)\right) H_z\right) < 1 \quad \forall z \in \partial \mathbf{D}, \tag{3}$$

where

$$G = \alpha^{-1} (D^{\sim})^{-1} \hat{G} D^{-1} \in \mathcal{G}_{\mathrm{L}_\infty},$$

and where, given $z \in \partial \mathbf{D}$, H_z is any matrix satisfying

$$H_z H_z^* = (I + G(z)^* G(z))^{-1}.$$

In view of (1), H_z can be selected as $\mathrm{ncfd}(G)(z)$ and (3) thus yields

$$\left\| \alpha^{-1} D P D^{-1} \mathrm{ncfd}(G) - \mathrm{ncfn}(G) \right\|_\infty < 1,$$

where we have used the fact that $G\mathrm{ncfd}(G) = \mathrm{ncfn}(G)$. Thus the right-hand side of (2) is no larger (actually, by continuity, smaller) than α. It follows that it is no larger than its left-hand side, and the proof is complete. □

Remark 1 *The right-hand side of (2) remains unaffected if either or both of the following relaxations are introduced: (i) G is allowed to have poles on $\partial \mathbf{D}$, i.e., G ranges over all real-rational transfer functions taking values* *in $\mathcal{G}$ almost everywhere on $\partial\mathbf{D}$; this is consistent with the discussion in [1]; (ii) the coprimeness restriction for the factorization of G is lifted, i.e., the ordered pair* $(\mathrm{ncfn(G)}, \mathrm{ncfd(G)})$ *is replaced by a pair of transfer functions ranging over the set*

$$\{(A,B) \in \mathbf{RH}_\infty \times \mathbf{RH}_\infty : A = \mathrm{ncfn}(G)U, B = \mathrm{ncfd}(G)U, U^{\sim}U = I\}.$$

The former follows from a simple continuity argument, the latter from the fact that coprimeness was not used in the proofs.

For the complex-μ case, we have the following corollary whose statement (for the continuous-time case) can be found, e.g., in Section 11.4 of [3].

Corollary 1 *Let $P \in \mathbf{RH}_\infty$ and let $k_\mathrm{r} = 0$. Then*

$$\sup_{z\in\partial\mathbf{D}} \hat{\mu}(P(z)) = \inf_{D\in\mathcal{D}^{\mathrm{U}}_{\mathrm{H}_\infty}} \left\|DPD^{-1}\right\|_\infty .$$

Remark 2 *Theorem 1 as well as the arguments used to prove it apply identically to the continuous-time case, except for obvious changes, such as replacing $\mathbf{D}$ with the open left half of the complex plane, and $\partial\mathbf{D}$ with the extended imaginary axis.*

The (D,G)-K iteration can be viewed as an attempt at minimizing the right-hand side of (2) over the set $\mathcal{K}$ of stabilizing controllers, i.e., at solving

$$\begin{array}{ll} \text{minimize } \beta & \\ \text{subject to} & E(K,\beta,D,G) := \left\| \left[\beta^{-1}DP(K)D^{-1} \quad -I\right] \begin{bmatrix} \mathrm{ncfd}(G) \\ \mathrm{ncfn}(G) \end{bmatrix} \right\|_\infty \leq 1, \\ & K \in \mathcal{K}, \beta > 0, D \in \mathcal{D}^{\mathrm{U}}_{\mathrm{H}_\infty}, G \in \mathcal{G}_{\mathrm{L}_\infty}, \end{array} \tag{4}$$

where $P(K)$ denotes the transfer function seen by the perturbation when controller K is in the loop, by means of a partially coordinate-wise iteration. Specifically, with reference to the step numbering used in Section 18.2 of [2], (i) in Step 4 of the (D,G)-K iteration, E is minimized with respect to K, while D, G, and β are kept fixed; (ii) in Step 5, the overall problem (4) is solved with respect to D, G, and β, while K is kept fixed, and (iii) in Step 6, E is minimized with respect to D and G, while K and β are kept fixed. Steps 5 and 6 are effected via the solution of LMIs at each point of a frequency grid (GEVPs in Step 5, EVPs in Step 6; see [8]), followed in the case of Step 6 with transfer function fitting and factorization (performed in Steps 2 and 3).

Note that components $\tilde{D}$ and $\tilde{G}$ of the minimizer in Step 5 are highly nonunique; a specific choice $(\hat{D}, \hat{G})$ is selected in Step 6. Arguably, Step 6 could be skipped and $(\tilde{D}, \tilde{G})$ used instead. Perhaps more astutely, use of a frequency grid could be dispensed with at Step 5 and the minimizer β_1, i.e., $\sup_{z \in \partial \mathbf{D}} \hat{\mu}(P(K)(z))$, computed using the algorithm proposed in [9].

References

[1] P.M. Young. Controller design with real parametric uncertainty. *Internat. J. Control*, 65:469–509, 1996.

[2] K. Zhou with J. C. Doyle. *Essentials of Robust Control.* Prentice Hall, New Jersey, 1998.

[3] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice Hall, New Jersey, 1996.

[4] Y.-S. Chou, A.L. Tits, and V. Balakrishnan. Stability multipliers and μ upper bounds: Connections and implications for numerical verification of frequency domain conditions. *IEEE Trans. Aut. Control*, 44(5):906–913, 1999.

[5] K. Poolla and A. Tikku. Robust performance against time-varying structured perturbations. *IEEE Transactions on Automatic Control*, AC-40(9):1589–1602, 1995.

[6] M. Vidyasagar. *Control System Synthesis. A Factorization Approach.* The MIT Press, Cambridge, Massachusetts, 1985.

[7] B. A. Francis. *A course in* $\mathbf{H}_\infty$ *Control Theory*, volume 88 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, 1987.

[8] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, June 1994.

[9] C. T. Lawrence, A. L. Tits, and P. Van Dooren. A fast algorithm for the computation of an upper bound on the μ-norm. *Automatica*, 36(3), 2000, to appear.