Robustness under Bounded Uncertainty with Phase Information

A. L. Tits*
Dept. of Elec. Eng.
and Inst. for Syst. Res.
University of Maryland
College Park, MD 20742
USA

V. Balakrishnan
School of Elec. and Comp. Eng.
Purdue University
West Lafayette, IN 47907-1285
USA

L. Lee
Dept. of Elec. Eng.
National Sun Yat-Sen Univ.
Kaohsiung, Taiwan 80424
R.O.C.

February 10, 1998

Abstract

We consider uncertain linear systems where the uncertainties, in addition to being bounded, also satisfy constraints on their phase. In this context, we define the "phase-sensitive structured singular value" (PS-SSV) of a matrix, and show that sufficient (and sometimes necessary) conditions for stability of such uncertain linear systems can be rewritten as conditions involving PS-SSV. We then derive upper bounds for PS-SSV, computable via convex optimization. We extend these results to the case where the uncertainties are structured (diagonal or block-diagonal, for instance).

1 Introduction

A popular paradigm for modeling control systems with uncertainties is illustrated in Fig. 1. Here $P(s)$ is the transfer function of a stable linear system, and Δ is a stable operator that represents the "uncertainties" that arise from various sources such as modeling errors, neglected or unmodeled dynamics or parameters, etc. Such control system models have found wide acceptance in robust control; see for example [1, 2, 3, 4].

From the physical laws governing the system and from the modeling procedures used to arrive at the paradigm in Fig. 1, the uncertainty Δ is usually known or assumed to possess various additional properties. Common examples are that Δ is structured (i.e., diagonal or block-diagonal), that it is linear time-invariant or real-constant etc. Often, information about the size of Δ (usually as a bound on some induced norm) is available. For example, if Δ is LTI, frequency response measurements can be used to estimate bounds on the $\mathbf{L}_2$ gain of Δ. It is also natural in some situations to assume that Δ is dissipative or passive, i.e., that it always dissipates energy. Such can be the case, for example, with high order mechanical systems when the dynamics associated with a (poorly known) passive subsystem (e.g., containing no energy sources and whose input and output are power-conjugate) are

*Research supported in part by NSF's Engineering Research Center Program, under grant NSFD-CDR-88-03012. The work of the third author was also supported in part by the National Science Council of Taiwan, R.O.C., under grant NSC-87-2218-E-110-004.

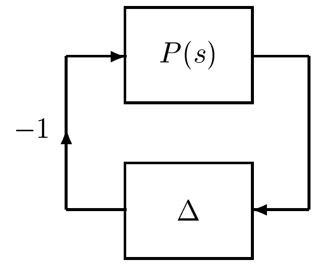


Figure 1: Closed-loop system

left out of the nominal model. For example, it is mentioned in [5, 6] that a good model for lightly damped flexible structures with colocated force actuators and rate sensors can be obtained in the paradigm in Fig. 1, where $P(s)$ is the transfer function of the nominal system model based on a few lower modal frequencies and mode shapes, with the higher modal frequencies and the corresponding mode shapes lumped together as a passive LTI uncertainty $\Delta(s)$. (Also see Example 1 in §5.)

Perhaps the most fundamental question that arises with the feedback system in Fig. 1 is that of "robust stability": Is the system stable for all possible instances of Δ? This question can be partially answered using a number of approaches. For example, if the $\mathbf{L}_2$-gain of Δ does not exceed one, the small gain theorem [7] asserts that the system is robustly stable if the $\mathbf{L}_2$ gain (which is also the $\mathbf{H}_\infty$ norm) of P is less than one. And, if Δ is known to be passive, the passivity theorem [7] asserts that closed-loop stability is ensured as long as P is strictly passive. If it is known further that Δ is block diagonal, then it is enough that an appropriately scaled version of P have $\mathbf{L}_2$-gain less than one or be strictly passive, respectively. Necessary and sufficient conditions in this context can be expressed in terms of the structured singular value (see for example [8]). Suppose now that Δ is known to be passive and, at the same time, to have an $\mathbf{L}_2$-gain no larger than one. Of course, either the small gain theorem or the passivity theorem can guarantee robust stability in this case, but intuitively, either approach alone would be too conservative, since in either case a seemingly important attribute of the uncertainty model is being ignored. One objective of this paper is to address this issue.

It is often the case that the uncertainty is linear, time-invariant (LTI), and diagonal. In this case, the unity-bounded $\mathbf{L}_2$-gain and passivity assumptions on Δ can be interpreted as knowledge on the frequency response of each diagonal entry δ_{ii} of Δ: the Nyquist plot of δ_{ii} lies inside the unit-disk and in the right-half complex plane, respectively. There are instances where it is appropriate to model δ_{ii} as having its Nyquist plot entirely contained within some acute sector, of aperture $2\theta < \pi$. Such a sector can be assumed, without loss of generality (via simple loop transformation), to be a proper subset of the right-half plane. This can occur when modeling is done from experimental data and the "Nyquist cloud" is better approximated by a sector portion of a disk than by a full disk; see Example 3 in §5. It can also occur when the uncertainty, due to several uncertain parameters, is "lumped" into a single dynamic uncertainty block; in this case, the approach presented in this paper would result in significant computational savings in comparison with the direct approach; see

Example 2 in §5. In both instances conservativeness can often be further reduced by allowing for frequency dependent sectors. Investigation of robust stability under such "uncertainty with phase information" is a further objective of this paper.

Uncertainty is often best represented by a set of full matrices (or block-diagonal matrices), and handling this situation in our framework necessitates a concept of "phase of a matrix". Several authors have proposed such concepts. In [9], the "principal phases" of a matrix are defined as the arguments of the eigenvalues of the unitary part of its polar decomposition, and a "small phase theorem" is derived that holds under rather stringent conditions. Hung and MacFarlane, in [10], propose a "quasi-Nyquist decomposition" in which the phase information of a transfer matrix is obtained by minimizing a measure of misalignment between the input and output singular vectors. Finally, Owens, in [11], uses the numerical range to characterize phase uncertainty in multivariable systems. The concept of phase we adopt here is related to that of [11]. Our definition not only serves to characterize phase uncertainty in multivariable systems, but also provides a practical and tractable way of using uncertainty phase information in robustness analysis.

Thus, in this paper, we consider the robust stability of the system in Fig. 1 when Δ is a block-diagonal LTI uncertainty that simultaneously satisfies constraints on its norm, and on its "phase". In §2, we define the phase-sensitive structured singular value (PS-SSV), defining in the process the phase of a matrix. We then derive a condition for robust stability of the system in Fig. 1 in terms of the PS-SSV. It turns out that when the uncertainty is scalar, or made of diagonal scalar blocks, the PS-SSV-based condition on robust stability is both necessary and sufficient. Computing the PS-SSV exactly turns out to be an NP-hard problem. We therefore concentrate on computing an upper bound on the PS-SSV, in §3. In §4, we show that computation of this upper bound can be reformulated as a quasi-convex optimization problem; we discuss some schemes for its solution. In §5, we demonstrate our results via three numerical examples, and we conclude with §6. Many of the ideas developed in this paper were adapted from earlier work by two of the coauthors and M. K. H. Fan, see [12, 13, 14, 15]. Results closely related to those of §3.1 were obtained independently by Eszter and Hollot [16] for the case when the phase bounds amount to a passivity constraint on the uncertainty.

Notation. $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_e$ denote the sets of real numbers, nonnegative real numbers, and $\mathbb{R} \cup \{\infty\}$ (one point-compactification of $\mathbb{R}$), respectively. $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_{+e}$ denote the set of complex numbers, complex numbers with positive real part (i.e., the open right-half complex plane), and $\mathbb{C}_+ \cup \{\infty\}$ respectively. $\mathbf{H}_\infty$ denotes the set of scalar- or matrix-valued functions that are analytic and bounded in the open right half plane, and $\mathbf{RH}_\infty$ denotes the set of functions in $\mathbf{H}_\infty$ that are real rational. H^* denotes the complex-conjugate transpose of $H \in \mathbb{C}^{m\times n}$. For $H \in \mathbb{C}^{n\times n}$ satisfying $H = H^*$, the notation $H \geq 0$ ($H > 0$) means that H is positive-semidefinite (positive-definite).

2 The phase-sensitive structured singular value

2.1 Scalar Case

Let us first consider the case when both the LTI system and the LTI uncertainty in Fig. 1 have a single input and a single output. Thus $P(s)$ and $\Delta(s)$ are scalar transfer functions, and to emphasize this, we rename them as $p(s)$ and $\delta(s)$, respectively.

Let $\phi : \mathbb{C} \to (-\pi, \pi]$ be the usual phase of a complex scalar, with $\phi(0)$ defined to be 0. Note that $|\phi(\cdot)|$ is lower semicontinuous (and continuous outside every neighborhood of the origin). Given a complex scalar m and a real scalar $\theta \in [0, \pi]$, let $\mu_\theta(m)$ be defined by

$$\mu_\theta(m) = \left(\inf\{|\gamma| \ : \ |\phi(\gamma)| \leq \theta, \ 1 + \gamma m = 0\}\right)^{-1},$$

if the set over which the infimum is taken is nonempty, and $\mu_\theta(m) = 0$ otherwise (i.e., $\mu_\theta(m) = |m|$ if $|\phi(m)| \geq \pi - \theta$, and 0 otherwise). Note that $\mu_\theta(m)$ is upper semicontinuous in (θ, m).

Theorem 1 below shows that various properties of the closed-loop system depicted in Fig. 1 can be assessed from the knowledge of $\mu_\theta(p(s))$ on the imaginary axis, under various assumptions on δ.

Theorem 1 *Let $p \in \mathbf{H}_\infty$ be continuous over $\overline{\mathbb{C}_{+e}}$, let $\theta : \mathbb{R}_e \to [0, \pi]$, and let*

$$\mathbf{\Delta}_\theta = \{\delta \in \mathbf{H}_\infty : \ \delta \text{ is continuous on } \overline{\mathbb{C}_{+e}}, \ \|\delta\|_\infty \leq 1, \ |\phi(\delta(\mathrm{j}\omega))| \leq \theta(\omega) \quad \forall \omega \in \mathbb{R}_e\}.$$

Suppose that
(a) $\sup_{\omega \in \mathbb{R}_e} \mu_{\theta(\omega)}(p(\mathrm{j}\omega)) < 1$.
Then $(1 + \delta p)^{-1} \in \mathbf{H}_\infty$ *for all* $\delta \in \mathbf{\Delta}_\theta$ *and, if* θ *is upper semicontinuous,*

$$\sup_{\delta \in \mathbf{\Delta}_\theta} \|(1 + \delta p)^{-1}\|_\infty < \infty. \tag{1}$$

Moreover, if θ is constant, then statements (a), (b) and (c) are equivalent, where (b) and (c) are as follows:
(b) $(1 + \delta p)^{-1} \in \mathbf{H}_\infty$ *for all* $\delta \in \mathbf{\Delta}_\theta$ *and* $\sup_{\delta \in \mathbf{\Delta}_\theta} \|(1 + \delta p)^{-1}\|_\infty < \infty$.
(c) $(1 + \delta p)^{-1} \in \mathbf{H}_\infty$ *for all* $\delta \in \mathbf{RH}_\infty \cap \mathbf{\Delta}_\theta$ *and* $\sup_{\delta \in \mathbf{RH}_\infty \cap \mathbf{\Delta}_\theta} \|(1 + \delta p)^{-1}\|_\infty < \infty$.

Proof: We first prove by contradiction that (a) implies that $(1 + \delta p)^{-1} \in \mathbf{H}_\infty$ for all $\delta \in \mathbf{\Delta}_\theta$. Thus suppose that, for some $\delta \in \mathbf{\Delta}_\theta$, $(1 + \delta p)^{-1} \notin \mathbf{H}_\infty$. It follows from Cauchy's Principle of the Argument, using a simple homotopy (see, e.g., Lemma 1 in [17, 18] for details) that there exists $\alpha \in (0, 1]$ and $\hat{\omega} \in \mathbb{R}_e$ such that

$$1 + \alpha\delta(\mathrm{j}\hat{\omega})p(\mathrm{j}\hat{\omega}) = 0.$$

Since $\delta \in \mathbf{\Delta}_\theta$, it is clear that $|\alpha\delta(\mathrm{j}\hat{\omega})| \leq 1$ and $|\phi(\alpha\delta(\mathrm{j}\hat{\omega}))| \leq \theta(\hat{\omega})$. Thus $\mu_{\theta(\hat{\omega})}(p(\mathrm{j}\hat{\omega})) \geq 1$, a contradiction. To complete the proof of the first claim, suppose that θ is upper semicontinuous and, proceeding again by contradiction, suppose that $(1 + \delta p)^{-1} \in \mathbf{H}_\infty$ for all $\delta \in \mathbf{\Delta}_\theta$ but that, given any $\epsilon > 0$ there exist $\delta_\epsilon \in \Delta$ and $\omega_\epsilon \in \mathbb{R}$ such that

$$|1 + \delta_\epsilon(\mathrm{j}\omega_\epsilon)p(\mathrm{j}\omega_\epsilon)| < \epsilon.$$

Let $\gamma_\epsilon = \delta_\epsilon(\mathrm{j}\omega_\epsilon)$ and note that, since $\delta_\epsilon \in \Delta$,

$$|\phi(\gamma_\epsilon)| \leq \theta(\omega_\epsilon).$$

Since $|\gamma_\epsilon| \leq 1$ it follows from compactness of the complex unit disk, continuity of p on $\mathrm{j}\mathbb{R}_\mathrm{e}$, lower semicontinuity of $|\phi|$ and upper semicontinuity of θ that there exits $\hat{\gamma} \in \overline{\mathbb{C}_{+\mathrm{e}}}$ and $\hat{\omega} \in \mathbb{R}_\mathrm{e}$ such that $|\hat{\gamma}| \leq 1$, $|\phi(\hat{\gamma})| \leq \theta(\hat{\omega})$ and $1 + \hat{\gamma}p(\mathrm{j}\hat{\omega}) = 0$. Thus $\mu_{\theta(\hat{\omega})}(p(\mathrm{j}\hat{\omega})) \geq 1$, a contradiction. To complete the proof of the theorem, first note that the implication (b)⇒(c) holds trivially. Suppose now that θ is constant. The implication (a)⇒(b) has just been proven. It thus remains remains to show that (c)⇒(a). We again use contradiction. Thus assume that

$$\sup_{\omega \in \mathbb{R}_\mathrm{e}} \mu_\theta(p(\mathrm{j}\omega)) \geq 1.$$

We show that, given any $\epsilon > 0$, there exists $\hat{\delta} \in \mathbf{\Delta}_\theta \cap \mathbf{RH}_\infty$ and $\hat{\omega} \in \mathbb{R}_\mathrm{e}$ such that $|1+\hat{\delta}(\mathrm{j}\hat{\omega})p(\mathrm{j}\hat{\omega})| < \epsilon$, a contradiction. Let $\tilde{\omega} \in \mathbb{R}_\mathrm{e}$ be such that $\mu_\theta(p(\mathrm{j}\tilde{\omega})) \geq 1$ (since p is continuous on $\mathrm{j}\mathbb{R}_\mathrm{e}$ and $\mu_\theta(\cdot)$ is upper semicontinuous, such $\tilde{\omega}$ always exists). Thus, for some $\tilde{\gamma} \in \mathbb{C}_+$, with $|\tilde{\gamma}| \leq 1$ and $|\phi(\tilde{\gamma})| \leq \theta$, $1 + \tilde{\gamma}p(\mathrm{j}\tilde{\omega}) = 0$. Note that, if $\theta = 0$, the claim holds trivially (take $\hat{\delta}(s) = \tilde{\gamma}$ for all s); thus assume that $\theta > 0$. Since p is continuous on $\mathrm{j}\mathbb{R}_\mathrm{e}$, there exist $\hat{\omega} \in \mathbb{R} \setminus \{0\}$ and $\hat{\gamma} \in \{\gamma \in \mathbb{C}_+ : |\gamma| < 1, |\phi(\gamma)| < \theta\}$ such that $|1 + \hat{\gamma}p(\mathrm{j}\hat{\omega})| < \epsilon$. It is shown in Appendix A that, under these conditions, there exists $\hat{\delta} \in \mathbf{\Delta}_\theta \cap \mathbf{RH}_\infty$ such that $\hat{\delta}(\mathrm{j}\hat{\omega}) = \hat{\gamma}$. This completes the contradiction argument. □

Remark: The upper semicontinuity assumption on θ is indeed needed in order for (1) (uniform robust stability) to follow, as shown by the following example. Let $p(s) = 2(s+1)/(s+2)$. Let $\hat{\omega}$ be the frequency at which the phase of $p(\mathrm{j}\omega)$ is largest (in the first quadrant) and let $\hat{\phi}$ be its value. Define θ by $\theta(\hat{\omega}) = \pi/2$ and $\theta(\omega) = \pi - \hat{\phi}$ for all other ω. It is readily checked that $\mu_{\theta(\omega)}(p(\mathrm{j}\omega)) = 0$ for all ω but that $1 + p(\mathrm{j}\omega)\delta(\mathrm{j}\omega)$ can be made arbitrarily close to 0 in the neighborhood of $\hat{\omega}$.

⋄

Remark: The sufficiency part of Theorem 1 can be extended to handle more general uncertainty sets. See remark immediately following Theorem 3.

⋄

We leave open the question of necessity of condition (a) of Theorem 1 under relaxed assumptions on θ. It may be necessary, e.g., to require that $\theta(\omega)$ not approach zero too fast.

2.2 The matrix case with structure

2.2.1 Phase and phase-sensitive structured singular value

As a first step toward extending the results of §2.1 to matrix-valued P and Δ, we propose a concept of phase of a matrix.

Given a complex matrix ?, let $\mathcal{N}(?)$ be its numerical range, i.e.,

$$\mathcal{N}(?) = \{x^* ? x \;:\; x \in \partial B\} \subset \mathbb{C}$$

where $\partial B = \{x \in \mathbb{C}^n : \|x\|_2 = 1\}$ and $\|\cdot\|_2$ is the Euclidean norm. This set is known to be convex. The following definition is a slight modification of that used in [15].

Definition 1 *Let* $? \neq 0$ *be a complex square matrix such that* $0 \notin \mathrm{int}\mathcal{N}(?)$*. The* median phase $\mathrm{MP}(?)$ *of* $?$ *is the angle, with a range of* $(-\pi, \pi]$*, between the positive real axis and the ray bisecting the smallest sector containing* $\mathcal{N}(?)$*. The* phase spread $\mathrm{PS}(?)$ *of* $?$ *is half the angle of this sector (see Fig. 2). We define* $\mathrm{MP}(0)$ *and* $\mathrm{PS}(0)$ *to be* 0*.*

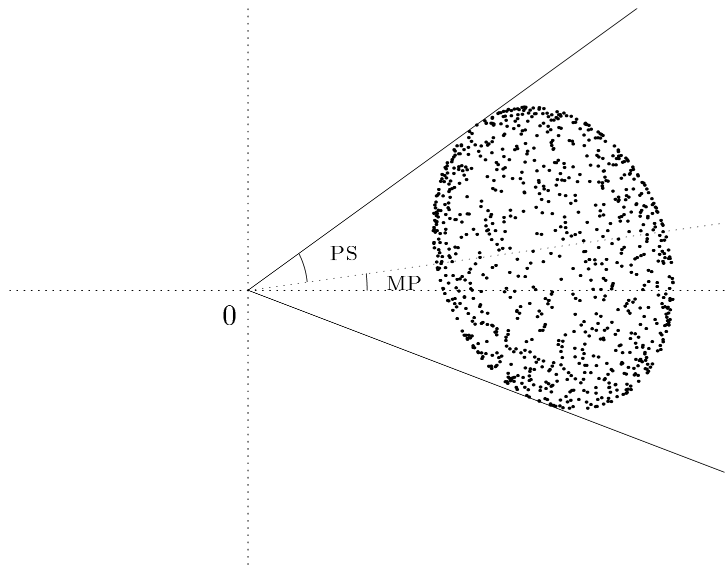


Figure 2: Numerical range, median phase and phase spread.

Thus $\mathrm{MP}(?) \in (-\pi, \pi]$ and $\mathrm{PS}(?) \in [0, \pi/2]$. Below we will refer to the pair $(\mathrm{MP}(?), \mathrm{PS}(?))$ as phase information of $?$. If $0 \in \mathrm{int}\mathcal{N}(?)$, there is no phase information for $?$.

Note that in the case of a complex number $a = \rho e^{j\phi}$ with $\rho > 0$ and $\phi \in (-\pi, \pi]$, the phase information of a is $(\phi, 0)$. Also, the phase information of a matrix is invariant under multiplication of the matrix by a positive number, and if $?$ is Hermitian positive semidefinite, its phase information is $(0, 0)$. Finally, the phase information of a matrix is invariant under unitary similarity transformations (since the numerical range is).

Median phase and phase spread are related to the concept of principal phases introduced by Postlethwaite et al. in [9]. Namely, for any square complex matrix $?$,

$$\mathrm{MP}(?) - \mathrm{PS}(?) \leq \psi_{\min}(?) \leq \psi_{\max}(?) \leq \mathrm{MP}(?) + \mathrm{PS}(?)$$

where $\psi_{\min}(?)$ and $\psi_{\max}(?)$ are the minimum and maximum principal phases of $?$, respectively. This result, stated differently, was obtained by Owens [11] (who also used the term "phase spread").

For any matrix $?$ with $0 \notin \mathrm{int}\mathcal{N}(?)$, $\mathcal{N}\left(e^{-\mathrm{jMP}(\Gamma)}?\right) \in \mathbb{C}_+$. In other words, we can rotate the numerical range of any matrix $?$ for which $0 \notin \mathrm{int}\mathcal{N}(?)$ so that it is contained in the right-half complex plane. With this in mind, we restrict our attention in the sequel to matrices $?$ with $? + ?^* \geq 0$ (or equivalently $\mathcal{N}(?) \subset \mathbb{C}_+$). For such matrices, we next give alternate characterizations of the phase information; these will serve us well in our derivation of stability tests in the sequel.

Given $?$ with $? + ?^* \geq 0$, of particular interest is the smallest sector (i.e, one that subtends the smallest angle at the origin) in the right-half plane, symmetric about the real axis, that contains $\mathcal{N}(?)$. (The interest stems from the fact that in the sequel, we will consider uncertainties Δ whose numerical range is known to lie in such symmetric sectors at every frequency.) Let $2\Phi(?)$ be the angle subtended by this sector at the origin. Evidently (see Fig. 2),

$$\Phi(?) = \max\left\{\mathrm{MP}(?) + \mathrm{PS}(?), -(\mathrm{MP}(?) - \mathrm{PS}(?))\right\}. \tag{2}$$

We then have the following alternate characterization for $\Phi(?)$.

Lemma 1 *Let* $? \in \mathbb{C}^{n\times n}$, *with* $? + ?^* \geq 0$. *Then,*

$$\Phi(?) = \cot^{-1}\left(\sup\left\{b: \ ? + ?^* - \frac{\beta}{\mathrm{j}}(? - ?^*) \geq 0 \ \forall\beta \in \{-b, b\}\right\}\right). \tag{3}$$

Proof: From Fig. 2 it is clear that, for $b > \cot(\Phi(?))$ (i.e., $b \neq 0$ and $b^{-1} < \tan(\Phi(?))$), there exists $\hat{v} \in \mathbb{C}^n$ such that

$$\mathrm{Re}(\hat{v}^* ? \hat{v}) < b|\mathrm{Im}(\hat{v}^* ? \hat{v})|$$

i.e., for some $\beta \in \{-b, b\}$,

$$? + ?^* - \frac{\beta}{\mathrm{j}}(? - ?^*) \not\geq 0.$$

Moreover, with $b = \cot(\Phi(?)) \in (-\infty, \infty)$, it is clear from the figure that, for all $v \in \mathbb{C}^n$,

$$\mathrm{Re}(v^* ? v) \geq \beta|\mathrm{Im}(v^* ? v)| \ \forall\beta \in \{-b, b\}$$

i.e.,

$$? + ?^* - \frac{\beta}{\mathrm{j}}(? - ?^*) \geq 0 \ \forall\beta \in \{-b, b\}.$$

Finally, if $\Phi(?) = 0$ then $\mathcal{N}(?)$ is a subset of the negative imaginary axis, i.e., $? = ?^* \geq 0$, and thus the matrix inequality in (3) holds for every finite β. □

Remark: It is easy to verify that for any Γ satisfying $\Gamma + \Gamma^* \geq 0$ with $\Phi(\Gamma) > 0$, we have

$$\Gamma + \Gamma^* - \frac{\beta}{\mathrm{j}}(\Gamma - \Gamma^*) \geq 0 \qquad \text{for all } \beta \in [-\cot\Phi(\Gamma), \cot\Phi(\Gamma)].$$

◇

Lemma 2 Φ *is lower semicontinuous over* $\{? : ? + ?^* \geq 0\}$.

Proof: Let $\theta \in [0, \pi/2]$ and let $\{?_k\} \to \hat{?}$, with $?_k + ?_k^* \geq 0$ for all k, and $\limsup \Phi(?_k) \leq \theta$. We show that $\Phi(\hat{?}) \leq \theta$, proving the claim. If $\theta = \pi/2$ the result is obvious. Thus suppose $\theta \in [0, \pi/2)$ and let $\epsilon \in (0, \pi/2 - \theta]$. For k large enough, $\Phi(?_k) \leq \theta + \epsilon$, and thus, taking the cotangent of both side of (3) ($\theta + \epsilon > 0$), for k large enough,

$$?_k + ?_k^* - \frac{\beta}{\mathrm{j}}(?_k - ?_k^*) \geq 0 \quad \forall\beta \in [-\cot(\theta + \epsilon), \cot(\theta + \epsilon)].$$

It follows that

$$\hat{?} + \hat{?}^* - \frac{\beta}{\mathrm{j}}(\hat{?} - \hat{?}^*) \geq 0 \quad \forall \beta \in [-\cot(\theta+\epsilon), \cot(\theta+\epsilon)],$$

i.e., (again using (3)), that $\cot(\Phi(\hat{?})) \geq \cot(\theta+\epsilon)$, i.e., $\Phi(\hat{?}) \leq \theta+\epsilon$. Since this holds for arbitrarily small $\epsilon > 0$, the claim follows. □

With an eye towards issues of robust stability with respect to possibly block-structured uncertainty, we now extend the definition of μ_θ to handle block-diagonal structures. Given positive integers k_i, $i = 1, \ldots, \ell$, such that $\sum k_i = n$, we define the set of block-diagonal matrices with block sizes k_i as

$$\mathbf{\Gamma} = \{\mathrm{diag}(?_1, \ldots, ?_\ell) : ?_i \in \mathbb{C}^{k_i \times k_i}\}. \tag{4}$$

We next define $\mathbf{\Gamma}_\Theta$ as the following phase-constrained subset of $\mathbf{\Gamma}$:

$$\mathbf{\Gamma}_\Theta = \{\mathrm{diag}(?_1, \ldots, ?_\ell) : ?_i \in \mathbb{C}^{k_i \times k_i},\ \Phi(?_i) \leq \theta_i\}, \tag{5}$$

where $\Theta = (\theta_1, \ldots, \theta_\ell)$ with, for $i = 1, \ldots, \ell$, $\theta_i \in [0, \pi/2]$. Note that $? + ?^* \geq 0$ for all $? \in \mathbf{\Gamma}_\Theta$.

Definition 2 *The phase-sensitive structured singular value of $M \in \mathbb{C}^{n \times n}$ with respect to $\mathbf{\Gamma}_\Theta$ is given by*

$$\mu_{\mathbf{\Gamma}_\Theta}(M) = (\inf\{\overline{\sigma}(?) : ? \in \mathbf{\Gamma}_\Theta, \det(I + ?M) = 0\})^{-1}$$

if $\det(I + ?M) = 0$ *for some* $? \in \mathbf{\Gamma}_\Theta$*, and* $\mu_{\mathbf{\Gamma}_\Theta}(M) = 0$ *otherwise.*

2.2.2 Properties of $\mu_{\mathbf{\Gamma}_\Theta}$

Unlike the "standard" mixed μ, $\mu_{\mathbf{\Gamma}_\Theta}$ is clearly not invariant under change of sign of its argument. Thus, in particular, it is not always larger than the spectral radius ρ (complex μ) or the real spectral radius $\rho_{\mathbb{R}}$. On the other hand it is clear that

$$\rho_{\mathbb{R}\_}(M) \leq \mu_{\mathbf{\Gamma}_\Theta}(M) \leq \mu(M), \tag{6}$$

where, for any complex matrix M,

$$\rho_{\mathbb{R}\_}(M) = \max\{\lambda : -\lambda \text{ is a negative, real eigenvalue of } M\},$$

with $\rho_{\mathbb{R}\_}(M) = 0$ if M has no negative, real eigenvalues. This leads to the following easily derived characterization of $\mu_{\mathbf{\Gamma}_\Theta}$. (Note that $\rho_{\mathbb{R}\_}$ is upper semicontinuous, which justifies the "max".)

Theorem 2

$$\mu_{\mathbf{\Gamma}_\Theta}(M) = \max_{\Gamma \in \mathbf{\Gamma}_\Theta,\, \overline{\sigma}(\Gamma) \leq 1} \rho_{\mathbb{R}\_}(?M) = \max_{\Gamma \in \mathbf{\Gamma}_\Theta,\, \overline{\sigma}(\Gamma) \leq 1} \rho_{\mathbb{R}\_}(M?). \tag{7}$$

Like the standard mixed μ, $\mu_{\mathbf{\Gamma}_\Theta}$ is invariant under similarity scaling of its argument by matrices that commute with the elements of the uncertainty set, i.e., given any nonsingular matrix $D = \mathrm{diag}(d_1 I_{k_1}, \ldots, d_\ell I_{k_\ell})$,

$$\mu_{\mathbf{\Gamma}_\Theta}(M) = \mu_{\mathbf{\Gamma}_\Theta}(DMD^{-1}).$$

In general however, $\mu_{\mathbf{\Gamma}_\Theta}$ is clearly not invariant under pre- or post-multiplication of its argument by a unitary matrix in $\mathbf{\Gamma}_\Theta$.

Next, it is readily verified that $\mu_{\mathbf{\Gamma}_\Theta}(M)$ is monotonic nondecreasing in each of the components of Θ and, using lower semicontinuity of Φ (Lemma 2), that $\mu_{\mathbf{\Gamma}_\Theta}(M)$ is upper semicontinuous in (Θ, M). Finally, the following result holds.

Proposition 1 *Let $P \in \mathbf{H}_\infty$ be continuous over $\overline{\mathbb{C}_{+\mathrm{e}}}$, and let $\Theta \in [0, \pi/2]^\ell$. Then*

$$\sup_{\omega \in \mathbb{R}_\mathrm{e}} \mu_{\mathbf{\Gamma}_\Theta}(P(\mathrm{j}\omega)) = \sup_{s \in \overline{\mathbb{C}_{+\mathrm{e}}}} \mu_{\mathbf{\Gamma}_\Theta}(P(s)).$$

Proof: We show that the following statements are equivalent:
(a)

$$\sup_{\omega \in \mathbb{R}_e} \mu_{\mathbf{\Gamma}_\Theta}(P(\mathrm{j}\omega)) < 1,$$

(b)

$$(I + ?\,P)^{-1} \in \mathbf{H}_\infty \quad \forall ? \in \mathbf{\Gamma}_\Theta, \overline{\sigma}(?) \le 1,$$

(c)

$$\sup_{s \in \overline{\mathbb{C}_{+\mathrm{e}}}} \mu_{\mathbf{\Gamma}_\Theta}(P(s)) < 1.$$

Since $\mu_{\mathbf{\Gamma}_\Theta}$ is positive homogeneous, the claim then follows from the equivalence of (a) and (c). We first show by contradiction that (a)$\Rightarrow$(b). Thus let $\hat{?} \in \mathbf{\Gamma}_\Theta$, with $\overline{\sigma}(\hat{?}) \le 1$, be such that $(I + \hat{?}\,P)^{-1} \notin \mathbf{H}_\infty$. As in the proof of Theorem 1, it follows from Cauchy's Principle of the Argument that there exist $\alpha \in (0, 1]$ and $\hat{\omega} \in \mathbb{R}_\mathrm{e}$ such that

$$\det(I + \alpha \hat{?}\,P(\mathrm{j}\hat{\omega})) = 0.$$

Since $\alpha\hat{?} \in \mathbf{\Gamma}_\Theta$ and $\overline{\sigma}(\alpha\hat{?}) \le 1$, this implies that $\mu_{\mathbf{\Gamma}_\Theta}(P(\mathrm{j}\hat{\omega})) \ge 1$, a contradiction. Concerning the implication (b)$\Rightarrow$(c), if there exists $\hat{s} \in \overline{\mathbb{C}_{+\mathrm{e}}}$ be such that $\mu_{\mathbf{\Gamma}_\Theta}(P(\hat{s})) \ge 1$, then there exists $\hat{?} \in \mathbf{\Gamma}_\Theta$, with $\overline{\sigma}(\hat{?}) \le 1$, such that $\det(I + \hat{?}\,P(\hat{s})) = 0$, contradicting (b). Finally, the implication (c)$\Rightarrow$(a) holds trivially. □

2.2.3 The small-$\mu_{\mathbf{\Gamma}_\Theta}$ theorem

Given any $\Theta : \mathbb{R}_\mathrm{e} \to [0, \pi/2]^\ell$, we define

$$\mathbf{\Delta}_\Theta = \{\Delta \in \mathbf{H}_\infty : \ \Delta \text{ is continuous on } \overline{\mathbb{C}_{+\mathrm{e}}}, \|\Delta\|_\infty \le 1,\ \Delta(\mathrm{j}\omega) \in \mathbf{\Gamma}_{\Theta(\omega)} \quad \forall \omega \in \mathbb{R}_\mathrm{e}\}.$$

Theorem 3 *Let $P \in \mathbf{H}_\infty$ be continuous over $\overline{\mathbb{C}_{+\mathrm{e}}}$, let $\Theta : \mathbb{R}_\mathrm{e} \to [0, \pi/2]^\ell$. Suppose that*
(a) $\sup_{\omega \in \mathbb{R}_\mathrm{e}} \mu_{\mathbf{\Gamma}_{\Theta(\omega)}}(P(\mathrm{j}\omega)) < 1$.
Then $(I + \Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \mathbf{\Delta}_\Theta$, and if Θ is upper semicontinuous, then

$$\sup_{\delta \in \mathbf{\Delta}_\Theta} \|(I + \Delta P)^{-1}\|_\infty < \infty.$$

Moreover, if Θ is constant, then (a) is equivalent to
(b) $(I + \Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \mathbf{\Delta}_\Theta$ and $\sup_{\delta \in \mathbf{\Delta}_\Theta} \|(I + \Delta P)^{-1}\|_\infty < \infty$.

Proof: The implication (a)$\Rightarrow$(b) is proved as in Theorem 1 with $\det(I+\Delta P)$ replacing $1+\delta p$. Concerning the implication (b)$\Rightarrow$(a), note that, if θ is constant and (a) does not hold, then (since P is continuous over $\overline{\mathbb{C}_{+e}}$ and $\mu_{\mathbf{\Gamma}_\Theta}$ is upper semicontinuous) there exists, among others, a constant (complex) $\Delta \in \mathbf{\Delta}_\Theta$ and some $\hat{\omega} \in \mathbb{R}_e$ such that $\det(I+\Delta P(\mathrm{j}\hat{\omega}))=0$, contradicting (b). □

Remark: Again, the sufficiency part of Theorem 3 can be extended to handle more general uncertainty sets. For example, consider the uncertainty set

$$\tilde{\mathbf{\Delta}}_\Theta = \{\Delta \in \mathbf{H}_\infty : \ \Delta \text{ is continuous on } \overline{\mathbb{C}_{+e}},\ U(\omega)\Delta(\mathrm{j}\omega) \in \mathbf{\Gamma}_{\Theta(\omega)},\ \overline{\sigma}(\Delta_i(\mathrm{j}\omega)) \le d_i(\omega), i=1,\dots,\ell, \quad \forall \omega \in \mathbb{R}_e\},$$

where $d_i : \mathbb{R}_e \to [0,\infty)$, $i=1,\dots,\ell$, $\Theta : \mathbb{R}_e \to [0,\pi/2]^\ell$, and $U(\omega)=\mathrm{diag}(u_1(\omega)I_{k_1},\dots,u_\ell(\omega)I_{k_\ell})$, with $u_i : \mathbb{R}_e \to \{z \in \mathbb{C} : |z|=1\}$. Then, it is easy to show that if

$$\sup_{\omega \in \mathbb{R}_e} \mu_{\mathbf{\Gamma}_{\Theta(\omega)}}(\mathrm{diag}(d_i(\omega)I_{k_i})U(\omega)^* P(\mathrm{j}\omega)) < 1,$$

then, $(1+\Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \tilde{\mathbf{\Delta}}_\Theta$; and that if, in addition, d_i and Θ are upper semicontinuous, and U is continuous, then

$$\sup_{\Delta \in \tilde{\mathbf{\Delta}}_\Theta} \|(1+\Delta P)^{-1}\| < \infty.$$

◇

Again we will leave open the question of necessity of condition (a) of Theorem 3 under relaxed assumptions on Θ. On the other hand, even for constant Θ, it is unclear in general whether, if (a) does not hold, there exists $\Delta \in \mathbf{\Delta}_\Theta$ real on the real axis (which must be the case if Δ is the transfer function of a real impulse response) such that $(I+\Delta P)^{-1} \notin \mathbf{H}_\infty$ or $\|(I+\Delta P)^{-1}\|_\infty$ is arbitrarily large. In the case of purely diagonal uncertainty structures, though, this is the case even with the additional requirement that Δ be rational. In other words the following holds.

Theorem 4 *Let $P \in \mathbf{H}_\infty$ be continuous over $\overline{\mathbb{C}_{+e}}$, let $\Theta \in [0,\pi/2]^\ell$, and suppose $k_i = 1$, $i=1,\dots,\ell$ $(=n)$, i.e., suppose that $\mathbf{\Gamma}_\Theta$ consists of diagonal matrices. The following statements are equivalent.*
(a) $\sup_{\omega \in \mathbb{R}_e} \mu_{\mathbf{\Gamma}_\Theta}(P(\mathrm{j}\omega)) < 1$.
(b) $(I+\Delta P)^{-1} \in \mathbf{H}_\infty$ *for all* $\Delta \in \mathbf{\Delta}_\Theta$ *and* $\sup_{\Delta \in \mathbf{\Delta}_\Theta} \|(I+\Delta P)^{-1}\|_\infty < \infty$.
(c) $(I+\Delta P)^{-1} \in \mathbf{H}_\infty$ *for all* $\Delta \in \mathbf{RH}_\infty \cap \mathbf{\Delta}_\Theta$ *and* $\sup_{\Delta \in \mathbf{RH}_\infty \cap \mathbf{\Delta}_\Theta} \|(I+\Delta P)^{-1}\|_\infty < \infty$.

(The proof of the implication (c)$\Rightarrow$(a) is exactly along the lines of that of the corresponding implication in Theorem 1.)

3 Upper Bounds on $\mu_{\mathbf{\Gamma}_\Theta}$

So far, we have seen definitions of $\mu_{\mathbf{\Gamma}_\Theta}$, and how conditions on $\mu_{\mathbf{\Gamma}_\Theta}$ give sufficient (and sometimes necessary) conditions for uniform robust stability. In this section, we will concern ourselves with the numerical computation of $\mu_{\mathbf{\Gamma}_\Theta}$.

Computing $\mu_{\mathbf{\Gamma}_\Theta}$ exactly is equivalent to finding the global minimum of a nonconvex optimization problem, and we are not aware of any efficient solution methods for it. Therefore, we will not attempt to compute $\mu_{\mathbf{\Gamma}_\Theta}$ directly; instead, we will derive numerically computable upper bounds on $\mu_{\mathbf{\Gamma}_\Theta}$, which will give, in turn, sufficient conditions for robust stability.

3.1 The matrix case with structure

Computing $\mu_{\mathbf{\Gamma}_\Theta}(m)$ for a scalar m is trivial. We then consider the problem of computing an upper bound on $\mu_{\mathbf{\Gamma}_\Theta}(M)$, when M is a matrix, and ? is assumed to have some structure, that is, it is required to belong to the set $\mathbf{\Gamma}_\Theta$. Let

$$\{S \ : \ S = \{\mathrm{diag}(s_1 I_{k_1}, \ldots, s_\ell I_{k_\ell}) : s_i > 0\}\,,$$

and, given $\Theta = (\theta_1, \ldots, \theta_\ell)$ with $\theta_i \in [0, \pi/2]$, $i = 1, \ldots, \ell$, let

$$\mathcal{B}_\Theta \triangleq \left\{ B \ : \ \begin{array}{l} B = \mathrm{diag}(\beta_1 I_{k_1}, \ldots, \beta_\ell I_{k_\ell}),\ \beta_i \in \mathbb{R},\ i = 1, \ldots, \ell, \\ \text{with } \beta_i \in [-\cot\theta_i, \cot\theta_i] \text{ when } \theta_i > 0 \end{array} \right\}.$$

We then have the following lemmas.

Lemma 3 *Let ? $\in \mathbf{\Gamma}_\Theta$, with $?^*? \leq I$. Then, ? satisfies, for every R, $S \in \mathcal{S}$, and $B \in \mathcal{B}_\Theta$,*

$$\begin{bmatrix} I \\ ? \end{bmatrix}^* \begin{bmatrix} R & (I - \mathrm{j}B)S \\ S(I + \mathrm{j}B) & -R \end{bmatrix} \begin{bmatrix} I \\ ? \end{bmatrix} \geq 0. \tag{8}$$

Proof: Consider any ? $\in \mathbf{\Gamma}_\Theta$ satisfying $?^*? \leq I$. Since ? commutes with every $R \in \mathcal{R}$, we have

$$R - ?^*R? \geq 0. \tag{9}$$

Next, since ? $\in \mathbf{\Gamma}_\Theta$, in view of Lemma 1 and of the remark following it, and since $-B \in \mathcal{B}_\Theta$ whenever $B \in \mathcal{B}_\Theta$, we have for every $B \in \mathcal{B}_\Theta$,

$$? + ?^* - \mathrm{j}(B? - ?^*B) \geq 0.$$

Since ?, every $S \in \mathcal{R}$ and every $B \in \mathcal{B}$ commute with each other, we then have

$$S? + ?^*S - \mathrm{j}(BS? - ?^*SB) \geq 0. \tag{10}$$

From (9) and (10), we conclude that every ? $\in \mathbf{\Gamma}_\Theta$ with $?^*? \leq I$ satisfies, for every R, $S \in \mathcal{S}$ and $B \in \mathcal{B}_\Theta$,

$$R - ?^*R? + (I - \mathrm{j}B)S? + ?^*S(I + \mathrm{j}B) \geq 0,$$

which is equivalent to (8). □

Theorem 5 *Let ? $\in \mathbf{\Gamma}_\Theta$, with $?^*? \leq I$. If there exists some $R \in \mathcal{S}$, $S \in \mathcal{S}$, and $B \in \mathcal{B}_\Theta$, such that*

$$M^*RM - R - (S(I + \mathrm{j}B)M + M^*(I - \mathrm{j}B)S) < 0, \tag{11}$$

then $\det(I + ?M) \neq 0$.

Remark: Theorem 5 constitutes a special case of the general stability theorem for systems with uncertainties described by integral quadratic constraints or IQCs [19, Theorem 1]. In particular, Theorem 5 can be viewed as a sufficient condition for the well-posedness of a feedback interconnection of a *constant* matrix with a *constant* phase- and norm-bounded uncertainty in the feedback loop. Since there are no dynamics involved, a direct linear algebraic proof can be given, which we present next for the sake of completeness.

$\diamond$

Proof: Rewriting (11) as

$$\begin{bmatrix} -M \\ I \end{bmatrix}^* \begin{bmatrix} R & (I - \mathrm{j}B)S \\ S(I + \mathrm{j}B) & -R \end{bmatrix} \begin{bmatrix} -M \\ I \end{bmatrix} < 0, \tag{12}$$

we now proceed by contradiction. Suppose that $\det(I + ?M) = 0$. Then for some nonzero $v \in \mathbb{C}^n$, we have $(I + ?M)v = 0$. Defining $u = Mv$, we have $v = -?u$. Now, from (12), we have

$$v^* \begin{bmatrix} -M \\ I \end{bmatrix}^* \begin{bmatrix} R & (I - \mathrm{j}B)S \\ S(I + \mathrm{j}B) & -R \end{bmatrix} \begin{bmatrix} -M \\ I \end{bmatrix} v < 0,$$

i.e.,

$$\begin{bmatrix} -u \\ v \end{bmatrix}^* \begin{bmatrix} R & (I - \mathrm{j}B)S \\ S(I + \mathrm{j}B) & -R \end{bmatrix} \begin{bmatrix} -u \\ v \end{bmatrix} < 0.$$

But from Lemma 3, we must have

$$\begin{bmatrix} I \\ ? \end{bmatrix}^* \begin{bmatrix} R & (I - \mathrm{j}B)S \\ S(I + \mathrm{j}B)S & -R \end{bmatrix} \begin{bmatrix} I \\ ? \end{bmatrix} \geq 0,$$

which yields

$$(-u)^* \begin{bmatrix} I \\ ? \end{bmatrix}^* \begin{bmatrix} R & (I - \mathrm{j}B)S \\ S(I + \mathrm{j}B) & -R \end{bmatrix} \begin{bmatrix} I \\ ? \end{bmatrix} (-u) \geq 0,$$

i.e.,

$$\begin{bmatrix} -u \\ v \end{bmatrix}^* \begin{bmatrix} R & (I - \mathrm{j}B)S \\ S(I + \mathrm{j}B) & -R \end{bmatrix} \begin{bmatrix} -u \\ v \end{bmatrix} \geq 0,$$

which is a contradiction. $\square$

We can use Theorem 5 to derive an upper bound $\hat{\mu}_{\mathbf{\Gamma}_\Theta}(M)$ on $\mu_{\mathbf{\Gamma}_\Theta}(M)$. Suppose that for some $\gamma > 0$, R and S in $\mathcal{S}$, and some B in $\mathcal{B}_\Theta$, we have

$$M^* R M - \gamma^2 R - (S(I + \mathrm{j}B)M + M^*(I - \mathrm{j}B)S) < 0.$$

Then, it can be shown with a little algebra that γ is an upper bound on $\mu_{\mathbf{\Gamma}_\Theta}(M)$. We therefore have the following upper bound on $\mu_{\mathbf{\Gamma}_\Theta}(M)$.[1]

Corollary 1 *Let $M \in \mathbb{C}^{n \times n}$. Then $\mu_{\mathbf{\Gamma}_\Theta}(M) \leq \hat{\mu}_{\mathbf{\Gamma}_\Theta}(M)$, where*

$$\hat{\mu}_{\mathbf{\Gamma}_\Theta}(M) = \inf \left\{ \gamma : \begin{array}{l} M^* R M - \gamma^2 R - S(I + \mathrm{j}B)M - M^*(I - \mathrm{j}B)S < 0 \\ \gamma > 0,\ R,\ S \in \mathcal{S},\ B \in \mathcal{B}_\Theta \end{array} \right\}. \tag{13}$$

[1] This result was first reported, in a slightly different form, in [12]. For the case when $\theta_i = \pi/2$, $i = 1, \ldots, \ell$ (passive uncertainty), it is a special case of a result obtained independently by Eszter and Hollot [16].

The conclusion of Corollary 1 represents one of the central contributions of the paper—we now have an upper bound for $\mu_{\mathbf{\Gamma}_\Theta}$, which, as we shall see in §4, can be numerically computed quite efficiently, using convex optimization techniques.

Remark: It is easily shown that, for any scalar m, $\hat{\mu}_{\mathbf{\Gamma}_\Theta}(m) = \mu_{\mathbf{\Gamma}_\Theta}(m)$.

$\diamond$

3.2 An off-axis circle-criterion interpretation

As was done in [20] and in §V of [21] in the context of the "classical" mixed μ, it is possible to obtain the phase-sensitive μ upper bound by optimizing the complex μ upper bound over a set of disk uncertainties. Consider a "block-diagonal disk uncertainty set", i.e., a set of block diagonal matrices such that each block ranges over a certain "hyperdisk", namely over the image of $\{?_i : \overline{\sigma}(?_i) \leq 1\}$ under a certain linear fractional transformation. A "complex-μ" type upper bound is readily obtained corresponding to such uncertainty blocks. Clearly, if the uncertainty set covers $\{? \in \mathbf{\Gamma}_\Theta : \overline{\sigma}(?) \leq 1\}$, then this upper bound is also an upper bound for $\mu_{\mathbf{\Gamma}_\Theta}$. Below we show that minimizing this bound over a certain family of such transformations yields precisely the bound given by Theorem 5 and Corollary 1.

Given $S \in \mathcal{S}$ and $B \in \mathcal{B}_\Theta$, let

$$T \triangleq \begin{bmatrix} F(I+F^*F)^{-1/2} & I \\ (I+F^*F)^{-1/2} & 0 \end{bmatrix},$$

where $F = S(I + \mathrm{j}B)$. It is readily checked that the "lower" linear fractional transformation $F_l(T, -M)$ is well defined for any M, that the "upper" linear fractional transformation $F_u(T, ?)$ is well defined whenever $\overline{\sigma}(?) \leq 1$, and that (provided $\overline{\sigma}(?) \leq 1$)

$$F_l(T, -M) = (F - M)(I + F^*F)^{-1/2},$$

$$F_u(T, ?) = ((I + F^*F)^{1/2} - ?F)^{-1}?.$$

Consequently, the systems in the three block diagrams of Fig. 3 are all equivalent in the sense that each one is well-posed if and only if the other two are.

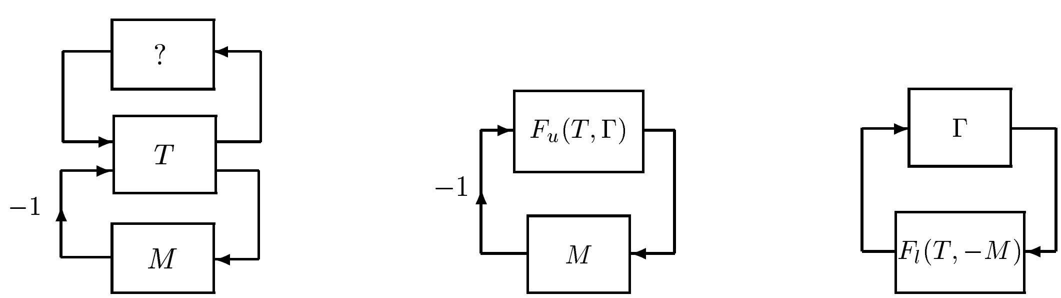


Figure 3: Three "equivalent" block diagrams.

For the sake of geometric intuition consider now the case of a diagonal (rather than block diagonal) structure, say, $F = \text{diag}(f_i)$, with $f_i = s_i(1 + \text{j}\beta_i)$, $s_i > 0$ and $|\beta_i| \leq \cot\theta_i$ when $\theta_i > 0$. Let $\mathbf{B\Gamma}$ be the set of complex diagonal matrices ? with $\overline{\sigma}(?) \leq 1$. The image of $\mathbf{B\Gamma}$ under the linear fractional transformation $F_u(T, \cdot)$ is given by

$$F_u(T, \mathbf{B\Gamma}) = \left\{ \text{diag}\left(\left(\gamma^{-1}\sqrt{1 + |f_i|^2} - f_i \right)^{-1} \right) : |\gamma| \leq 1 \right\}.$$

It is straightforward to check that each diagonal entry ranges over a circle of radius $\sqrt{1 + |f_i|^2} = \sqrt{1 + s_i^2(1 + \beta_i^2)}$ centered at $\overline{f}_i = s_i(1 - \text{j}\beta_i)$ (see Fig. 4). It follows that, for each $s_i > 0$ and each β_i with $|\beta_i| \leq \cot\theta_i$ when $\theta_i > 0$,

$$\{? \in \mathbf{\Gamma}_\Theta : \overline{\sigma}(?) \leq 1\} \subset F_u(T, \mathbf{B\Gamma}), \tag{14}$$

which shows that each diagonal "disk" entry of $F_u(T, \mathbf{B\Gamma})$ "covers" the corresponding entry in the uncertainty set of interest, $\{? \in \mathbf{\Gamma}_\Theta : \overline{\sigma}(?) \leq 1\}$. Conversely, it is easy to show that any disk that covers a diagonal entry in the uncertainty set $\{? \in \mathbf{\Gamma}_\Theta : \overline{\sigma}(?) \leq 1\}$ must be the corresponding diagonal entry of $F_u(T, \mathbf{B\Gamma})$ for some T: it is easy to solve "backwards" for s_i and β_i, given the center and radius of the disk.

The same inclusion (14) holds in the general (block diagonal) case. Indeed

$$T^{-1} = \begin{bmatrix} 0 & (I + F^*F)^{1/2} \\ I & -F \end{bmatrix}.$$

and simple algebra shows that, for any ? with $\overline{\sigma}(?) \leq 1$, $F_l(T^{-1}, ?)$ is well defined and

$$F_u(T, F_l(T^{-1}, ?)) = ?,$$

and thus it is enough to show that

$$F_l(T^{-1}, \{? \in \mathbf{\Gamma}_\Theta : \overline{\sigma}(?) \leq 1\}) \subset \mathbf{B\Gamma}.$$

To see that the latter inclusion holds, assume without loss of generality that $\ell = 1$ (full matrix uncertainty), i.e., $F = fI$, with $f = s(1 + \text{j}\beta)$, $s > 0$, $|\beta| \leq \cot\theta$ when $\theta > 0$, and let $? \in \mathbf{\Gamma}_\Theta$ with $\overline{\sigma}(?) \leq 1$. It remains to show that

$$\overline{\sigma}(F_l(T^{-1}, ?)) \leq 1,$$

or, equivalently,

$$I - (1 + |f|^2)(I + \bar{f}?^*)^{-1}?^*?(I + f?)^{-1} \geq 0,$$

i.e., via a congruence transformation,

$$(I + \bar{f}?^*)(I + f?) - (1 + |f|^2)?^*? \geq 0,$$

i.e.

$$(I - ?^*?) + (f? + \bar{f}?^*) \geq 0.$$

Since this clearly holds for any $? \in \mathbf{\Gamma}_\Theta$ with $\overline{\sigma}(?) \leq 1$, (14) holds in the general case as claimed.

A sufficient condition for

$$\det(I + ?M) \neq 0 \quad \forall ? \in \mathbf{\Gamma}_\Theta,\ \overline{\sigma}(?) \leq 1,$$

i.e., for $\mu_{\mathbf{\Gamma}_\Theta}(M) < 1$, is thus that

$$\det(I + ?M) \neq 0 \quad \forall ? \in F_u(T, \mathbf{B\Gamma}).$$

Since the second and third block diagrams in Fig. 3 are equivalent, the latter holds if and only if

$$\det(I - F_l(T, -M)?) \neq 0 \quad \forall ? \in \mathbf{B\Gamma},$$

i.e.,

$$\mu(F_l(T, -M)) < 1,$$

and a sufficient condition for this is that, for some $R \in \mathcal{S}$,

$$\overline{\sigma}(RF_l(T, -M)R^{-1}) < 1. \tag{15}$$

Since S and B commute, letting $M_R = RMR^{-1}$, we get

$$RF_l(T, -M)R^{-1} = (F - M_R)(I + F^*F)^{-1/2}.$$

It follows that (15) holds if, and only if

$$((F - M_R)(I + F^*F)^{-1/2})^*((F - M_R)(I + F^*F)^{-1/2}) < I,$$

i.e., if, and only if,

$$(F^* - M_R^*)(F - M_R) < I + F^*F,$$

which holds if, and only if,

$$M_R^*M_R - I - F^*M_R - M_R^*F < 0,$$

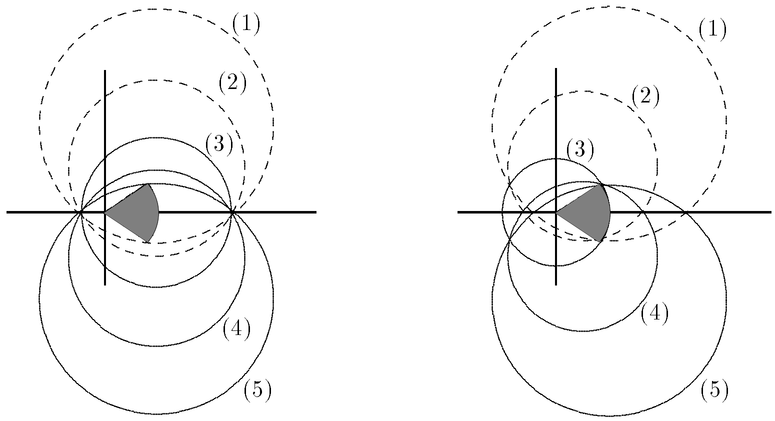
which is equivalent to the condition given in Theorem 5.

3.3 Some special cases

It is instructive to study the application of Theorem 5 and Corollary 1 to some special cases for the set $\mathbf{\Gamma}_\Theta$. These cases are encountered more often in practice; also, for some of these special cases, we can relate our results to those from literature.

3.3.1 Bounded passive uncertainty

We consider first the case when the $\mathbf{\Gamma}_\Theta$ consists of unstructured or full matrices (i.e., $\ell = 1$) with a known bound on their maximum singular value, and whose phase is known to be $\pi/2$ or less. This situation arises when the uncertainty Δ is *passive* and *bounded*. If Δ were scalar (i.e., $k_1 = 1$), this



(1): $\beta_i = -\cot(\theta_i)$
(2): $-\cot(\theta_i) < \beta_i < 0$
(3): $\beta_i = 0$
(4): $0 < \beta_i < \cot(\theta_i)$
(5): $\beta_i = \cot(\theta_i)$

(1),(2): $\beta_i = -\cot(\theta_i)$
(4),(5): $\beta_i = \cot(\theta_i)$
(1),(5): $s_i = 1$
(2),(4): $0 < s_i < 1$
(3): $s_i = 0$

Figure 4: Covering the uncertainty with off-axis circles. The figure on the left shows the covering of the ith diagonal phase-bounded uncertainty with disk-uncertainties obtained by loop transforming the unit-disk with $f_i = (1 + \mathrm{j}\beta_i)$ for various values of β_i. The figure on the right shows the covering with disk-uncertainties obtained by loop transforming the unit-disk with $f_i = s_i(1 \pm \mathrm{j}\cot\theta_i)$ for various values of s_i.

would mean that the Nyquist plot of Δ is in a semicircle of known radius that lies in the right-half complex plane, shown in in Fig. 5(a).

In this case, $\mathcal{B}_\Theta = \{0\}$ and $\mathcal{S}$ consists of positive multiples of the identity matrix. Therefore, from Corollary 1, we have

$$\hat{\mu}_{\mathbf{r}_\Theta}(M) = \inf\left\{\gamma : \begin{array}{c} rM^*M - \gamma^2 rI - s\,(M + M^*) < 0 \\ \gamma > 0,\ r > 0,\ s > 0 \end{array}\right\},$$

which further simplifies to

$$\hat{\mu}_{\mathbf{r}_\Theta}(M) = \left(\max\left\{0,\ \inf\left\{\lambda_{\max}\left(M^*M - c\,(M + M^*)\right)\ :\ c > 0\right\}\right\}\right)^{1/2},$$

where $\lambda_{\max}$ denotes the largest eigenvalue of the corresponding Hermitian matrix.

3.3.2 Bounded, constant, Hermitian, positive-definite uncertainty

We next consider the case when Δ is a constant, Hermitian, positive-definite matrix, with a known bound on its maximum singular value. If the uncertainty were scalar (i.e., $k_1 = 1$), this would mean that the Nyquist plot of the uncertainty is simply a point in a sub-interval of the positive real axis, as shown in Fig. 5(b).

(a) Bounded passive uncertainty: The Nyquist plot is known to lie in the shaded region.

(b) Positive, real uncertainty: The Nyquist plot is known to lie in a sub-interval of the positive real axis.

Figure 5: Various special cases

In this case the set $\mathcal{B}_\Theta$ consists of arbitrary real multiples of the identity, while $\mathcal{S}$ consists of positive multiples of the identity. Therefore, we have,

$$\hat{\mu}_{\mathbf{\Gamma}_\Theta}(M) = \inf \left\{ \gamma : \begin{array}{l} rM^*M - \gamma^2 rI - s(1+\mathrm{j}b)M - M^*(1-\mathrm{j}b)s < 0 \\ \gamma > 0,\ r > 0,\ s > 0,\ b \in \mathbb{R} \end{array} \right\},$$

which further simplifies to

$$\hat{\mu}_{\mathbf{\Gamma}_\Theta}(M) = \left(\max\left\{0,\ \inf\left\{\lambda_{\max}\left(M^*M - (c+\mathrm{j}d)M - M^*(c-\mathrm{j}d)\right)\ :\ c > 0,\ d \in \mathbb{R}\right\}\right\}\right)^{1/2}.$$

It is instructive to consider other special cases of the instances considered above, when the uncertainty is *diagonal*, so that $k_1 = \cdots = k_\ell = 1$.

3.3.3 Diagonal bounded passive uncertainty

Suppose that the Nyquist plot of each of the diagonal uncertainties is known to lie a half-disk such as the one shown in Fig. 5(a). In other words the uncertainty is diagonal, passive and bounded. In this case, the set $\mathcal{B}_\Theta = \{0\}$ and the set $\mathcal{S}$ consists of diagonal positive-definite matrices. Here

$$\hat{\mu}_{\mathbf{\Gamma}_\Theta}(M) = \inf \left\{ \gamma : \begin{array}{l} M^*RM - \gamma^2 R - SM - M^*S < 0 \\ \gamma > 0,\ R,\ S > 0 \text{ and diagonal} \end{array} \right\}.$$

3.3.4 Diagonal, bounded, positive, constant real uncertainty

Finally, we consider the case when each of the diagonal uncertainties is a constant unknown parameter, known only to lie in some sub-interval of the positive real axis such as the one shown in Fig. 5(b). Such uncertainties are often called parametric uncertainties. Here, the set $\mathcal{B}_\Theta$ consists of arbitrary real diagonal matrices, while $\mathcal{S}$ consists of diagonal positive-definite matrices. Thus,

$$\hat{\mu}_{\mathbf{\Gamma}_\Theta}(M) = \inf \left\{ \gamma : \begin{array}{l} M^*RM - \gamma^2 R - S(I+\mathrm{j}B)M - M^*(I-\mathrm{j}B)S < 0 \\ \gamma > 0,\ R,\ S,\ B \text{ real and diagonal},\ R > 0,\ S > 0 \end{array} \right\}. \quad (16)$$

Remark: This case of bounded diagonal real uncertainty is well-studied in the literature, usually under the name of "real-μ" analysis; see for example, [21, 22]. The problem considered in these references is the computation of $\mu_{\mathbb{R}}(M)$, which is defined as

$$\mu_{\mathbb{R}}(M) \triangleq \begin{cases} \left(\inf\left\{\overline{\sigma}(\Gamma): \begin{array}{c}\Gamma \text{ is diagonal and real,} \\ \det(I+\Gamma M)=0\end{array}\right\}\right)^{-1} & \text{if } \det(I+\Gamma M)=0 \text{ for some diagonal and real } \Gamma, \\ 0 & \text{otherwise} \end{cases}$$

We point out that $\mu_{\mathbb{R}}(M)$ is different from $\mu_{\mathbf{\Gamma}_\Theta}(M)$ with $\Theta = 0$ (for ease of reference, we will call the latter quantity $\mu_{\mathbb{R}_+}$ and its upper bound given in (16) by $\hat{\mu}_{\mathbb{R}_+}$). The difference between $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{R}_+}$ is that with $\mu_{\mathbb{R}_+}$, the uncertainty is required to be nonnegative, unlike with the definition of $\mu_{\mathbb{R}}$. For this reason, we will refer to $\mu_{\mathbb{R}}$ as "two-sided real-μ", while we will call $\mu_{\mathbb{R}_+}$ "one-sided real-μ".

The upper bound for the two-sided real-μ from [21] and [22] can be easily adapted via a loop transformation to yield an upper bound for the one-sided real-μ. This upper bound on $\mu_{\mathbb{R}_+}$ is just

$$\tilde{\mu}_{\mathbb{R}_+}(M) = \inf\left\{\gamma: \begin{array}{l} -2\gamma S - S(I+\mathrm{j}B)M - M^*(I-\mathrm{j}B)S < 0 \\ \gamma > 0,\ S,\ B \text{ diagonal, } S > 0 \end{array}\right\}. \tag{17}$$

Remarkably, computing $\tilde{\mu}_{\mathbb{R}_+}$ using (17) has the same complexity as computing $\hat{\mu}_{\mathbb{R}_+}$ using (16). Extensive numerical simulations suggest that this upper bound is tighter than the bound (16). We should note however that the bound (17) *does not* extend to the case of general phase-bounded uncertainty considered in this paper.

Finally, we note that it is possible to adapt $\hat{\mu}_{\mathbb{R}_+}$, the upper bound for the one-sided real μ, to yield an upper bound for the two-sided real μ. This upper bound on $\mu_{\mathbb{R}}$ turns out to be

$$\inf\left\{\gamma: \begin{array}{l} M^*\left(3R+2S\right)M - \gamma^2 R + \gamma\left(\left(S-R+jB\right)M - M^*\left(2S-R-jB\right)\right) < 0 \\ \gamma > 0,\ S,\ B \text{ diagonal, } S > 0 \end{array}\right\}.$$

However, we know of no efficient way of computing this upper bound.

◇

4 Computing $\sup_\omega \hat{\mu}_{\mathbf{\Gamma}_{\Theta(\omega)}}(P(\mathrm{j}\omega))$

From Theorem 3 in §2.2.3, it should be clear that the computation of $\sup_{\omega\in\mathbb{R}_e} \mu_{\mathbf{\Gamma}_{\Theta(\omega)}}(P(\mathrm{j}\omega))$, which we shall denote by $\mathcal{M}_\Theta(P)$, is of considerable interest. For reasons pointed out at the beginning of §3, we will consider instead the problem of computing $\sup_\omega \hat{\mu}_{\mathbf{\Gamma}_{\Theta(\omega)}}(P(\mathrm{j}\omega))$, which we shall denote by $\hat{\mathcal{M}}_\Theta(P)$. Since $\hat{\mathcal{M}}_\Theta(P) \geq \mathcal{M}_\Theta(P)$, computing $\hat{\mathcal{M}}_\Theta(P)$ will enable us to state sufficient conditions for the stability of the system in Fig. 1.

For each frequency ω, the quantity $\hat{\mu}_{\mathbf{\Gamma}_{\Theta(\omega)}}(P(\mathrm{j}\omega))$, defined in Corollary 1, can be computed as the solution to a quasi-convex optimization problem. There are several ways of showing this; we will demonstrate one method. For convenience, we let $M = P(\mathrm{j}\omega)$.

Recall that $\hat{\mu}_{\mathbf{\Gamma}_\Theta}$ is given by (13). Let $T = BS$. Then the condition on B is equivalent to

$$\Lambda S > T > -\Lambda S,$$

where Λ is a constant diagonal matrix given by

$$\Lambda = \mathrm{diag}\left(\cot\left(\theta_1(\omega)\right) I_{k_1},\ \cdots, \cot\left(\theta_\ell(\omega)\right) I_{k_\ell}\right).$$

Thus $\hat{\mu}_{\mathbf{\Gamma}_\Theta}$ is given as the optimal value of γ obtained by solving the problem

$$\begin{array}{ll} \text{minimize} & \gamma^2 \\ \text{subject to} & \gamma^2 R > M^* R M - (SM + M^* S) - \mathrm{j}(TM - M^* T) \\ & \Lambda S > T > -\Lambda S, \\ & R = \mathrm{diag}(r_1 I_{k_1}, \dots, r_\ell I_{k_\ell}), \quad r_i > 0 \\ & S = \mathrm{diag}(s_1 I_{k_1}, \dots, s_\ell I_{k_\ell}), \quad s_i > 0 \\ & T = \mathrm{diag}(t_1 I_{k_1}, \dots, t_\ell I_{k_\ell}) \end{array} \tag{18}$$

With $\nu \triangleq \gamma^2$, the optimization variables in this problem are ν, R, S and T. Problem (18) is one of minimizing a linear objective ν, subject to constraints on ν, R, S and T that are convex (in fact, linear matrix inequalities[2]) in R, S and T for fixed ν, and vice versa. It can be shown that problem (18) is a quasi-convex optimization problem [23]. Much work has been done lately on problems such as (18): it is well-known that such problems have polynomial worst-case complexity; moreover, very efficient algorithms and software tools are available for their solution [24, 25].

Next, we have the following obvious lower bound on $\hat{\mathcal{M}}_\Theta(P)$.

Lemma 4 *Let $\Omega = \{\omega_0, \omega_1, \dots, \omega_N\}$ be a set of frequencies. Then, $\hat{\mathcal{M}}_\Theta^{\mathrm{lb}}(P,\Omega)$, defined as*

$$\hat{\mathcal{M}}_\Theta^{\mathrm{lb}}(P,\Omega) \triangleq \max_i \left\{ \hat{\mu}_{\mathbf{\Gamma}_{\Theta(\omega_i)}}(P(\mathrm{j}\omega_i)) \right\},$$

satisfies $\hat{\mathcal{M}}_\Theta^{\mathrm{lb}}(P,\Omega) \le \hat{\mathcal{M}}_\Theta(P)$, i.e., it is a lower bound on $\hat{\mathcal{M}}_\Theta(P)$.

In order to compute $\hat{\mathcal{M}}_\Theta^{\mathrm{lb}}(P,\Omega)$, we need to solve $N+1$ quasi-convex optimization problems of the form (18). Of course, the number and choice of frequencies comprising Ω determines how tight a bound $\hat{\mathcal{M}}_\Theta^{\mathrm{lb}}(P,\Omega)$ is.

Remark: The lower bound given by Lemma 4 suffers from a possible shortcoming: It is known that in general, $\hat{\mu}_{\mathbf{\Gamma}_{\Theta(\omega)}}(P(\mathrm{j}\omega))$ may be *discontinuous* as a function of ω. Specifically, $\hat{\mu}_{\mathbf{\Gamma}_{\Theta(\omega)}}(P(\mathrm{j}\omega))$ might only be upper semicontinuous, and therefore we have no guarantees with the convergence of the lower bound $\hat{\mathcal{M}}_\Theta^{\mathrm{lb}}(P,\Omega)$ to $\hat{\mathcal{M}}_\Theta(P)$ even if N, the number of elements of Ω, tends to ∞ (but a scheme analogous to that proposed in [26] might be applicable). However, in most engineering applications (as we will see in §5), this does not pose a serious problem.

It is also possible to compute upper bounds on $\hat{\mathcal{M}}_\Theta(P)$ using state-space methods. The basic idea is this. $\hat{\mathcal{M}}_\Theta(P) \le \gamma$ if and only if there exist $R : \mathrm{j}\mathbb{R} \to \mathbb{R}^{n\times n}$, $S : \mathrm{j}\mathbb{R} \to \mathbb{R}^{n\times n}$ and $T : \mathrm{j}\mathbb{R} \to \mathbb{R}^{n\times n}$ such that for

[2] A linear matrix inequality or an LMI is a matrix inequality of the form $F(x) = F_0 + \sum_{i=1}^m x_i F_i > 0$ or $F(x) \ge 0$, where F_i are given Hermitian matrices, and the x_is are the real optimization variables.

every $\omega \in \mathbb{R}_e$, the following constraints are satisfied (the dependence of Λ on ω is now made explicit).

$$
\begin{aligned}
&\text{(i)} && \gamma^2 R(\mathrm{j}\omega) > P(\mathrm{j}\omega)^* R(\mathrm{j}\omega) P(\mathrm{j}\omega) - (S(\mathrm{j}\omega)P(\mathrm{j}\omega) + P(\mathrm{j}\omega)^* S(\mathrm{j}\omega)) - \\
& && \qquad \mathrm{j}(T(\mathrm{j}\omega)P(\mathrm{j}\omega) - P(\mathrm{j}\omega)^* T(\mathrm{j}\omega)), \\
&\text{(ii)} && \Lambda(\omega) S(\mathrm{j}\omega) > T(\mathrm{j}\omega), \\
&\text{(iii)} && T(\mathrm{j}\omega) > -\Lambda(\omega) S(\mathrm{j}\omega), \\
&\text{(iv)} && R(\mathrm{j}\omega) = \mathrm{diag}(r_1(\mathrm{j}\omega) I_{k_1}, \ldots, r_\ell(\mathrm{j}\omega) I_{k_\ell}), \quad r_i(\mathrm{j}\omega) > 0 \\
&\text{(v)} && S(\mathrm{j}\omega) = \mathrm{diag}(s_1(\mathrm{j}\omega) I_{k_1}, \ldots, s_\ell(\mathrm{j}\omega) I_{k_\ell}), \quad s_i(\mathrm{j}\omega) > 0 \\
&\text{(vi)} && T(\mathrm{j}\omega) = \mathrm{diag}(t_1(\mathrm{j}\omega) I_{k_1}, \ldots, t_\ell(\mathrm{j}\omega) I_{k_\ell})
\end{aligned}
\tag{19}
$$

It can be shown [27] that the constraints in (19) hold for some γ if and only if they hold for some real-rational transfer functions $\hat{R}$, $\hat{S}$ and $\hat{T}$. This fact can be combined with the Positive-Real (PR) lemma [28, 29] to write down LMIs whose feasibility is equivalent to conditions (i)–(v) (see for example, [22, 27] for an illustration of this procedure). Thus, a sufficient condition for the feasibility of problem (19) can be recast as an LMI feasibility problem. A bisection scheme can then be used to compute an upper bound for $\hat{\mathcal{M}}_\Theta(P)$. It is also possible to avoid the bisection scheme altogether, by recasting the upper bound computation problem as a single generalized eigenvalue minimization problem; see [30].

$\diamond$

5 Numerical examples

We demonstrate on a few examples the application of stability tests based on the PS-SSV.

5.1 Example 1: Stability of a flexible structure

We consider the stability of a planar truss structure, with a model adapted from the one presented in [5]. The truss structure has sixteen free nodes, each with two degrees of freedom; thus it exhibits thirty-two flexible modes. We assume that the first mode is exactly modeled as a linear time-invariant system, with transfer function p given by

$$p(s) = \frac{0.3927^2 s^2}{s^2 + 2(0.0075)(131)s + 131^2}.$$

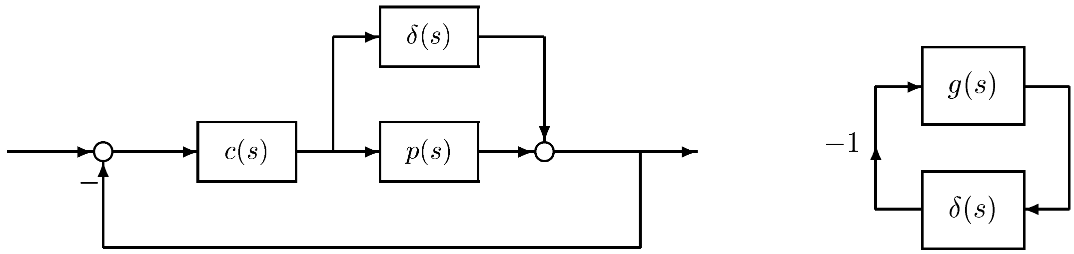
The remaining modes are modeled as a linear time-invariant uncertainty, with transfer function denoted by $\delta(s)$. It is known that δ is stable, and satisfies

$$|\delta(\mathrm{j}\omega)| \leq 0.3370, \quad \mathrm{Re}\,\delta(\mathrm{j}\omega) \geq 0, \text{ for all } \omega \in \mathbb{R}, \tag{20}$$

that is, δ is passive, and has an $\mathbf{H}_\infty$ norm bound of 0.3370. A linear time-invariant controller c with transfer function

$$c(s) = \frac{2.38s^5 + 33.18s^4 + 40842.00s^3 + 489341.01s^2 + 203926.51s + 489289.16}{s^5 + 15.15s^4 + 10927.81s^3 + 163193.36s^2 + 587196.79s + 434923.70}$$

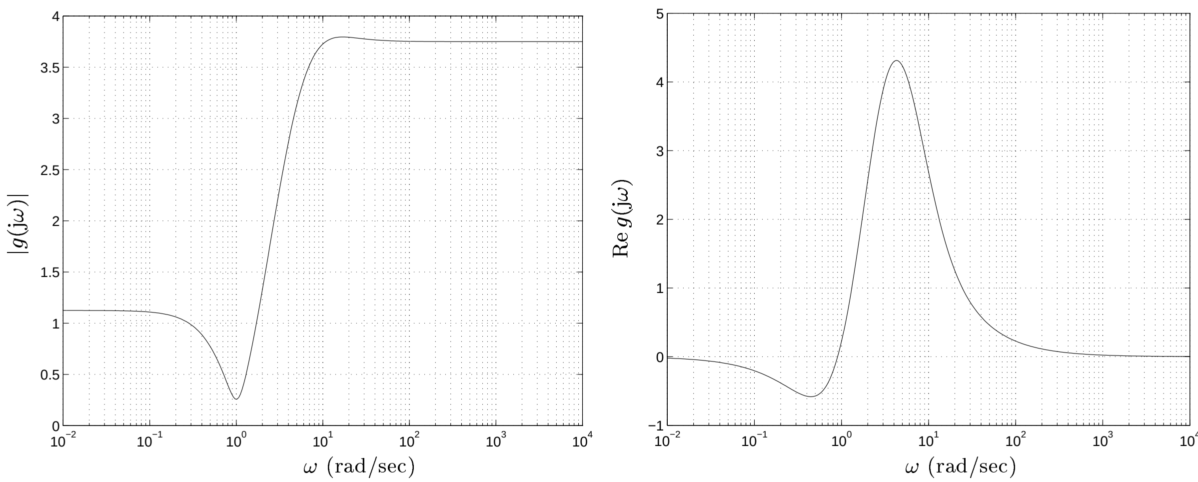
has been designed to stabilize $p(s)$, placing the poles at -1, -4 and -10. The robust stability question then is whether the controller stabilizes $p + \delta$.



(a) Block diagram of the flexible truss structure.

(b) Block diagram redrawn in our framework.

Figure 6: Example 1: Models of the flexible truss structure.



(a) Magnitude of g.

(b) Real part of g.

Figure 7: Example 1: Frequency response of g.

The block diagram of the system is shown in Fig. 6(a). The system redrawn in our analysis framework is shown in Fig. 6(b), where $g = c/(1+pc)$. The magnitude and real part of g are shown in Fig. 7.

From an inspection of these plots, and the properties of δ given in (20), we conclude that:

- The small gain theorem does not prove stability of the system in Fig. 6(b), since the $\mathbf{H}_\infty$ norm of g exceeds $1/0.3370$.
- The passivity theorem does not prove stability of the system in Fig. 6(b), since g is not strictly passive (the real part of $g(\mathrm{j}\omega)$ is nonpositive for some ω).

However, the analysis techniques presented in this paper do prove uniform robust stability. A plot of $\mu_{\mathbf{\Gamma}_\Theta}(g(\mathrm{j}\omega))$ is shown in Fig. 8. (Since g is a scalar transfer function, $\mu_{\mathbf{\Gamma}_\Theta}$ is trivial to compute.) Since $\sup_{\omega\in\mathbb{R}_e}\mu_{\mathbf{\Gamma}_\Theta}(g(\mathrm{j}\omega)) < 1/0.3370$, the system in Fig. 6(b) is indeed uniformly robustly stable.

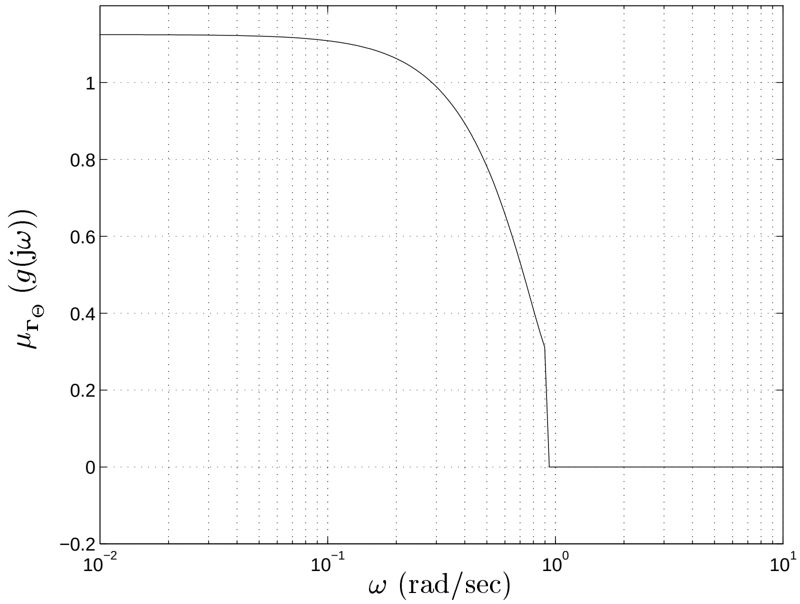


Figure 8: Example 1: The PS-SSV of $g(\mathrm{j}\omega)$ versus ω.

5.2 Example 2: Analysis of parametric systems

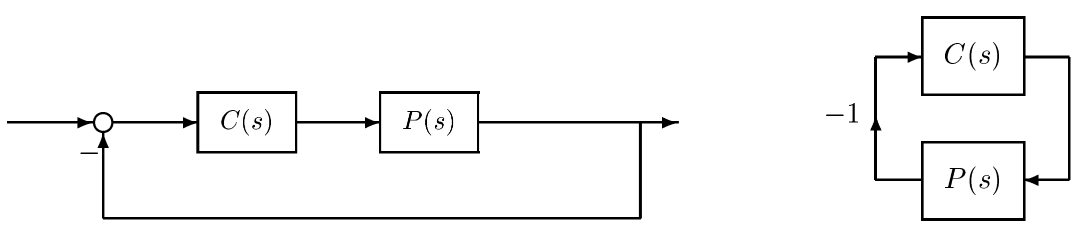
We next consider the problem of uniform robust stability of the closed-loop system shown in Fig. 9(a). P is the parameter-dependent plant, with transfer function given by

$$P(s) = \mathrm{diag}(p_1(s), p_2(s)), \quad p_i(s) = \frac{a_i(s+b_i)}{s^2+2c_is+1}, \quad a_i \in [0,1],\ b_i \in [1,2],\ c_i \in [1,2],$$

and C is the controller with the transfer function

$$C(s) = 0.3 \begin{bmatrix} \dfrac{s^2+s-1}{(s+1)(s+2)} & \dfrac{s+1}{s+2} \\ 1 & \dfrac{s+1}{s+10} \end{bmatrix}.$$

The problem now is to ascertain the stability of this system for all allowable values of the parameters.



(a) Block diagram of the parameter-dependent system.

(b) Block diagram redrawn in our framework.

Figure 9: Example 2: Stability analysis of a parameter-dependent system.

Fig. 10 shows the values of the frequency response of p_i, over a number of allowable parameter values, at a sample list of frequencies. This figure indicates that each p_i is passive, and has a frequency response which can be described as satisfying certain magnitude and phase constraints. Fig. 11 shows the magnitude and phase constraints on each of the terms.

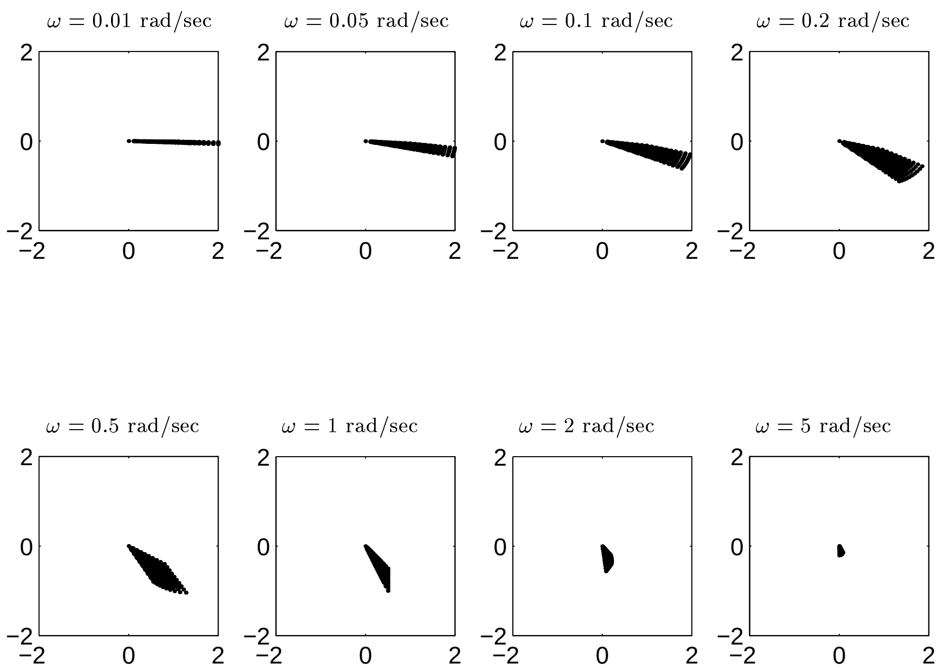
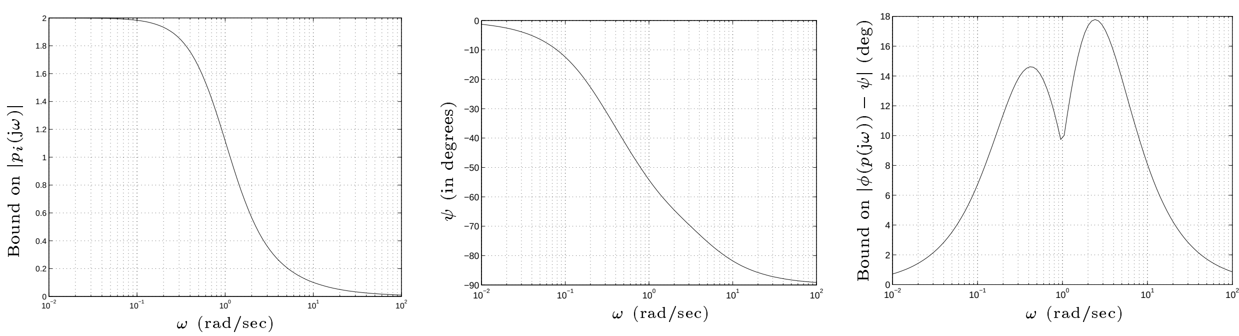


Figure 10: Example 2: Frequency response of each p_i at a number of frequencies.



(a) Magnitude constraints on $p_i(\mathrm{j}\omega)$.

(b) Average value of $\phi(p_i(\mathrm{j}\omega))$.

(c) Magnitude of variation of $\phi(p_i(\mathrm{j}\omega))$ about its average.

Figure 11: Example 2: Magnitude and phase constraints on $p_i(\mathrm{j}\omega)$.

This problem can be posed in our PS-SSV framework, as shown in Fig. 9(b). The uniform robust stability condition is

$$\sup_{\omega \in \mathbb{R}_e} \overline{\sigma}(P(\mathrm{j}\omega))\mu_{\mathbf{\Gamma}_{\Theta(\omega)}} \left(e^{\mathrm{j}\psi(\omega)} C(\mathrm{j}\omega) \right) < 1, \tag{21}$$

where $\psi(\omega)$ and the entries of $\Theta(\omega)$ are plotted against ω in Figs. 11(b) and 11(c), respectively. For convenience we let $\tilde{C}(\mathrm{j}\omega) = e^{\mathrm{j}\psi(\omega)}C(\mathrm{j}\omega)$. A plot of $\hat{\mu}_{\mathbf{\Gamma}_{\Theta(\omega)}}(\tilde{C}(\mathrm{j}\omega))$ is shown in Fig. 12, in solid lines. For reference, the optimally scaled maximum singular value of $\tilde{C}(\mathrm{j}\omega)$ is shown in dotted lines; this is an upper bound on $\mu(\tilde{C}(\mathrm{j}\omega))$, which can be thought of as an upper bound on PS-SSV that does not use the phase information. Since condition (21) holds, the system in Fig. 9(b) is indeed uniformly robustly stable. Note that, since $\overline{\sigma}(P(\mathrm{j}0)) = 2$, the bound on PS-SSV that does not use the phase information does not yield this conclusion.

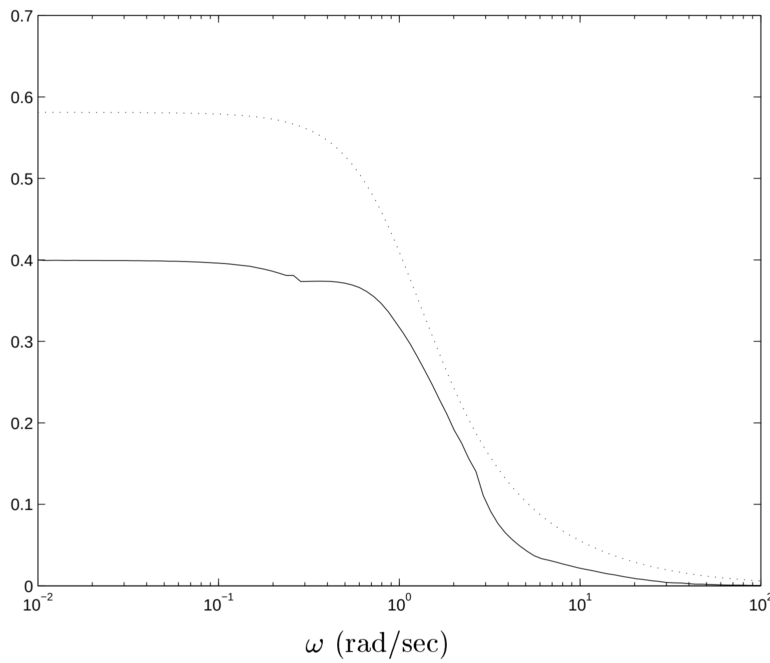


Figure 12: Example 2: The upper bound on $\mu_{\mathbf{\Gamma}_{\Theta(\omega)}}\left(\tilde{C}(\mathrm{j}\omega)\right)$ is plotted against ω in solid lines. The optimally scaled maximum singular value of $\tilde{C}(\mathrm{j}\omega)$ is plotted against ω in dotted lines.

Remark: There is a more direct method of analyzing parameter-dependent systems, namely "real-μ" analysis (see [21]). It is of interest to compare PS-SSV-based stability methods with real-μ methods.

Let us consider the question of whether the system in Fig. 9(b) is uniformly robustly stable. The answer is affirmative in the PS-SSV framework if $\sup_{\omega\in\mathbb{R}_e}\overline{\sigma}(P(\mathrm{j}\omega))\hat{\mu}_{\mathbf{\Gamma}_{\Theta(\omega)}}\left(\tilde{C}(\mathrm{j}\omega)\right)$ is less than one. Checking this numerically, from the discussion in §4 (in particular, Lemma 4), requires the solution of N LMI feasibility problems, one for each frequency. Let us consider one such feasibility problem. The variables in this problem are diagonal 2×2 matrices R, S and T. Thus, the number of scalar variables is 6. There is one LMI constraint of size 2×2, and 6 scalar constraints.

When the uniform robust robust stability of the same system is posed in the real-μ framework of [21], we once again have to solve an LMI feasibility problem at each frequency. Here the variables in each problem are diagonal 6×6 matrices $D = D^T$ and $G = G^T$ (see [21] for details); thus the number of scalar optimization variables is 12. There is one LMI constraint of size 6×6, and 6 scalar constraints.

For the problem of uniform robust stability with parametric uncertainties, PS-SSV-based tests are likely to be more conservative than real-μ tests. However, it should be clear from the number of variables and constraints that the amount of computation required by PS-SSV-based methods is less than that required by real-μ methods. For our example, empirical studies indicate that the computation required by real-

μ methods is approximately 12 times that required by PS-SSV-based methods [31].[3] Thus, the PS-SSV approach can be useful in analyzing parameter-dependent systems, albeit more conservatively, when the number of parameters is large.

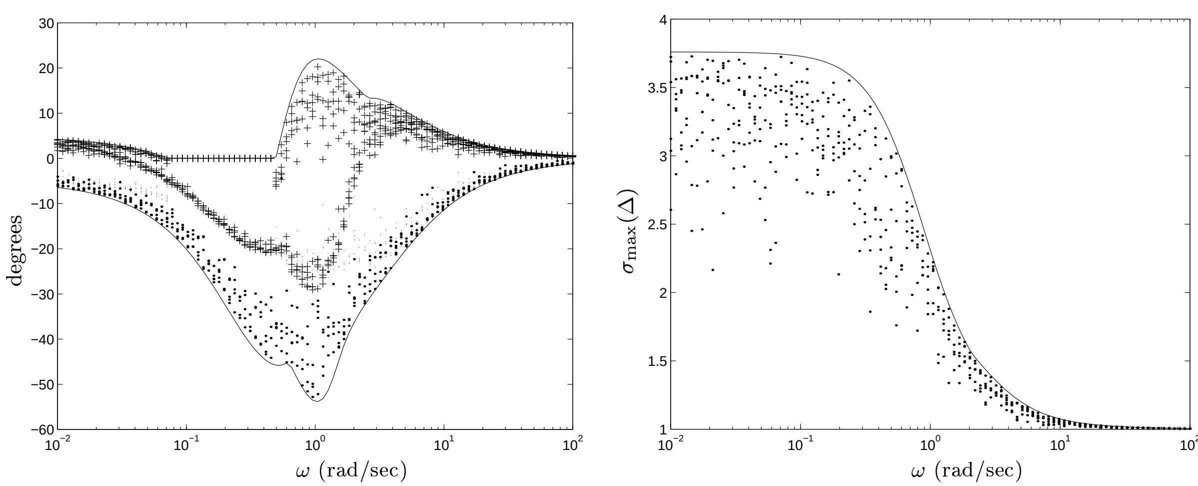
$\diamond$

5.3 Example 3: Experimentally measured matrix phase information

We consider an uncertain system as in Fig. 1, where the plant P is strictly proper (i.e, $P(\infty) = 0$), has two inputs, two outputs, and a state-space realization (A, B, C) with

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \frac{1}{7}\begin{bmatrix} -1 & 0 & 1 \\ 1 & 2 & -2 \end{bmatrix}.$$

We assume that the two-input two-output LTI uncertainty Δ has been experimentally measured at a number of frequencies. A scatter-plot of the phase information of $\Delta(\mathrm{j}\omega)$ at a number of frequencies is shown in Fig. 13(a); a scatter-plot of the norm of $\Delta(\mathrm{j}\omega)$ at a number of frequencies is shown in Fig. 13(b).



(a) Scatter plot of the $\mathrm{MP}(\Delta(\mathrm{j}\omega)) + \mathrm{PS}(\Delta(\mathrm{j}\omega))$ (denoted by "+") and $\mathrm{MP}(\Delta(\mathrm{j}\omega)) - \mathrm{PS}(\Delta(\mathrm{j}\omega))$ (denoted by "."), for various Δ, versus ω.

(b) Scatter plot of the norm of $\Delta(\mathrm{j}\omega)$ (for various Δ) versus ω.

Figure 13: Example 3: Experimentally determined magnitude and phase characteristics of Δ.

From the scatter plot shown in Fig. 13(a), we can determine continuous functions ϕ_{lb} and ϕ_{ub} such that for every frequency ω and Δ, the smallest sector containing $\mathcal{N}(\Delta(\mathrm{j}\omega))$ is

$$\left\{ z \ : \ z = re^{j\psi}, \ r \geq 0, \ \psi \in [\phi_{\mathrm{lb}}(\omega), \phi_{\mathrm{ub}}(\omega)] \right\}.$$

(These functions are shown in solid lines in Fig. 13(a).) Also, from Fig. 13(b), we can determine a function $d(\omega)$ such that for every frequency ω and Δ,

$$\overline{\sigma}(\Delta(\mathrm{j}\omega)) < d(\omega).$$

[3] In general, for an LMI problem with k variables and L LMI constraints of size $n_i \times n_i$, the computation required is dominated by $O\left(k^2 \sum_{i=1}^{L} n_i(n_i+1)/2\right)$.

(This function is shown in a solid line in Fig. 13(b).)

Then, defining $\theta(\omega) = 0.5(\phi_{\rm ub}(\omega) - \phi_{\rm lb}(\omega))$ and $\psi(\omega) = 0.5(\phi_{\rm ub}(\omega) + \phi_{\rm lb}(\omega))$, we have that the system in Fig. 1 is uniformly robustly stable if

$$\sup_{\omega \in \mathbb{R}_{\rm e}} \mu_{\mathbf{\Gamma}_{\Theta(\omega)}} \left(e^{{\rm j}\psi(\omega)} P({\rm j}\omega) \right) d(\omega) < 1,$$

where in the notation of §2.2.1, $k_1 = 2$, and $\Theta = (\theta)$. The upper bound $\hat{\mu}_{\mathbf{\Gamma}_{\Theta(\omega)}}(P({\rm j}\omega)e^{{\rm j}\psi(\omega)})$ from (13) is obtained for various ω by solving the optimization problem (18), and plotted in Fig. 14. Since $\sup_{\omega \in \mathbb{R}_{\rm e}} \hat{\mu}_{\mathbf{\Gamma}_{\Theta(\omega)}}(e^{{\rm j}\psi(\omega)} P({\rm j}\omega)) d(\omega) < 1$, the system in Fig. 1 is indeed uniformly robustly stable.

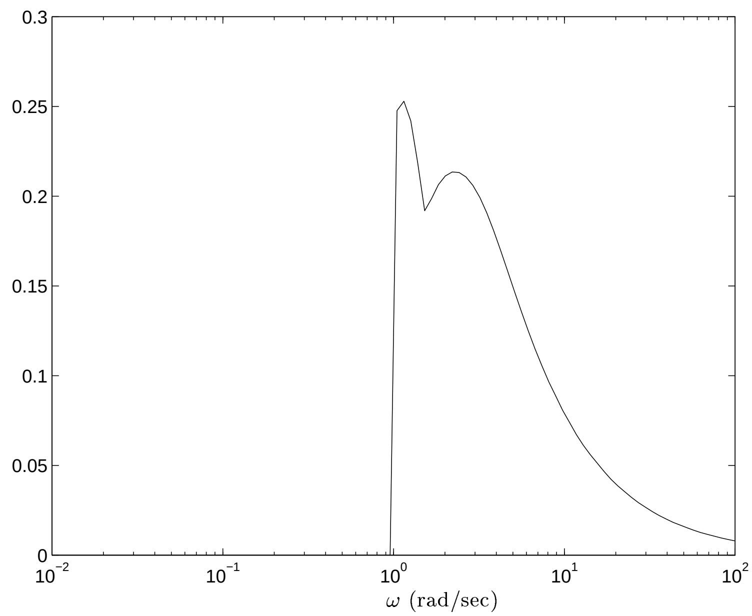


Figure 14: Example 3: Upper bound on $\mu_{\mathbf{\Gamma}_{\Theta(\omega)}}({\rm e}^{{\rm j}\psi(\omega)} P({\rm j}\omega))$ as a function of ω.

6 Conclusions

The "phase-sensitive structured singular value" framework developed in this paper provides an effective robustness analysis tool in various situations, e.g., in the case when the uncertainty, besides being (possibly block-structured and) small, is known to be passive.

Several issues have been left unresolved.

1. Under what "minimal" assumptions on $\theta(\cdot)$ are statements (a), (b) and (c) of Theorem 1 equivalent?
2. In the presence of non-scalar "full blocks", and with Θ constant, are statements (a) and (b) in Theorem 3 equivalent to the analogue of statement (c) in Theorem 1, namely

   (c) $(I + \Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \mathbf{RH}_\infty \cap \mathbf{\Delta}$ and $\sup_{\delta \in \mathbf{RH}_\infty \cap \mathbf{\Delta}} \|(I + \Delta P)^{-1}\|_\infty < \infty$.

3. When is the upper bound $\hat{\mu}_{\boldsymbol{\Gamma}_\Theta}$ defined in Corollary 1 of §3 equal to $\mu_{\boldsymbol{\Gamma}_\Theta}$, in particular is it always equal to $\mu_{\boldsymbol{\Gamma}_\Theta}$ when $\ell = 1$ (full block uncertainty)?

The answer to some of these questions may be within reach.

The contributions in the paper can be generally viewed as the following: When the uncertainty Δ in Fig. 1 is LTI, and when additional information on the phase of the frequency response of Δ is available, we have derived sufficient (and sometimes necessary) conditions for robust stability. A natural extension of this problem considered in this paper is the following. Consider for simplicity the case when Δ is a scalar uncertainty, and suppose that it is known that the Nyquist plot of Δ is restricted to lie in some region in the complex plane that can be described as the intersection of generalized disks (i.e., disks and half-spaces). Then, we can derive a sufficient robust stability condition by combining robust stability conditions for each generalized disk, just as we did to arrive at Theorem 5. As a further extension along these lines, consider the situation when the Nyquist plot of Δ is restricted to lie in some region in the complex plane that can be described as the union of sets which are themselves obtained as an intersection of generalized disks. (A classic example of such a region is the "butterfly" uncertainty set, described in [14].) The techniques described in this paper can be extended to handle these more general cases as well.

The focus of this paper has been exclusively on uncertainties about which phase information is available. The techniques herein can be combined with other standard robustness analysis techniques such as complex or real-μ analysis, when phase information about only certain blocks of the uncertainty is available, leading to a new "mixed-μ" paradigm. Finally, while the theory was developed for the continuous-time case, extension to discrete time is straightforward.

Appendix A

Proposition 2 *Let $\theta \in (0, \pi]$, let $\hat{\omega} \in \mathbb{R} \setminus \{0\}$, and let $\gamma \in \overline{\mathbb{C}_+}$ be such that $|\gamma| < 1$ and $|\phi(\gamma)| < \theta$. There exists $\delta \in \mathbf{RH}_\infty$, continuous on $\overline{\mathbb{C}_{+e}}$, such that $\delta(\mathrm{j}\hat{\omega}) = \gamma$ and such that $\|\delta\|_\infty < 1$ and $\sup_{\omega \in \mathbb{R}} |\phi(\delta((\mathrm{j}\omega)))| < \theta$.*

Proof: If $\gamma = 0$, simply let δ map $\overline{\mathbb{C}_+}$ to zero. Assume now $\gamma \neq 0$. Let $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{D}^\theta = \{z \in \mathcal{D} : \mathrm{Re}\, z \geq 0, |\phi(z)| < \theta\}$. We first construct a non-rational mapping $\tilde{\delta} : \overline{\mathcal{D}} \to \mathbb{C}$, taking real values on the real axis, such that $\tilde{\delta}(\overline{\mathcal{D}})$ belongs to $\mathcal{D}^\theta$ and contains γ and $1/2 + \mathrm{j}0$ in its interior. This map is selected from a one-parameter family of mappings $\tilde{\delta}^\lambda : \overline{\mathcal{D}} \to \mathbb{C}$, $\lambda \in (0, 1)$, constructed as the composition of two maps, i.e., $\tilde{\delta}^\lambda = \tilde{\delta_2}^\lambda \circ \tilde{\delta_1}^\lambda$.

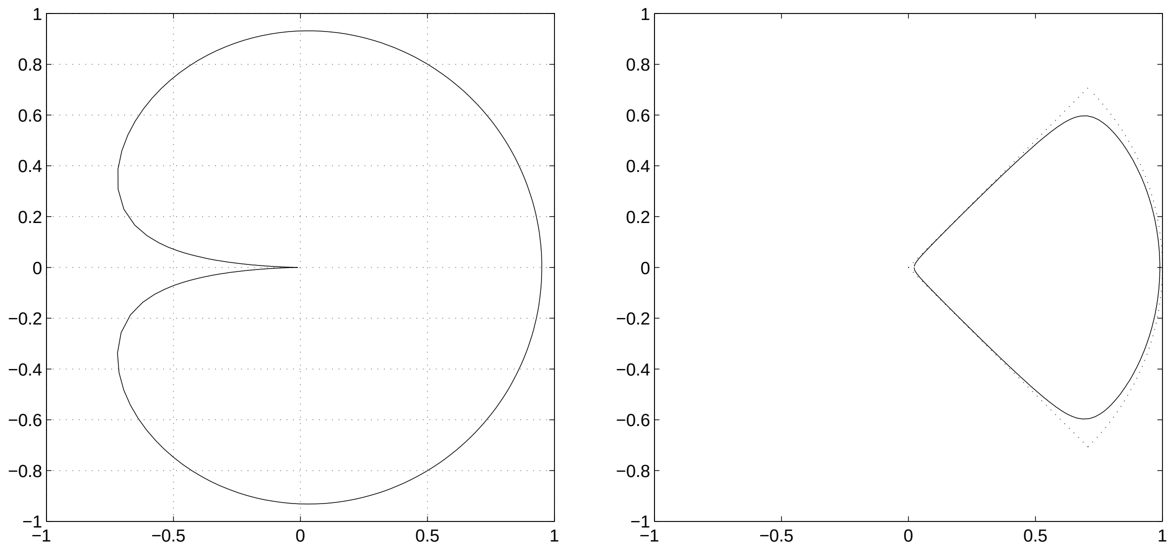
First, for $\lambda \in (0, 1)$, the map $\tilde{\delta_1}^\lambda$, defined on $\overline{\mathcal{D}}$, is given by

$$\tilde{\delta_1}^\lambda(z) = -\frac{1 + \lambda z - (1 - 2\lambda z \cos\psi + (\lambda z)^2)^{1/2}}{1 + \lambda z + (1 - 2\lambda z \cos\psi + (\lambda z)^2)^{1/2}}, \text{ with } \sin(\psi/2) = \frac{\lambda}{2 - \lambda}.$$

For fixed λ, $\tilde{\delta_1}^\lambda$ maps $\mathcal{D}$ to the interior of a set such as the one depicted in Fig. 15(a).

Next $\tilde{\delta_2}^\lambda$, defined on $\tilde{\delta_1}^\lambda(\overline{\mathcal{D}})$, is given by

$$\tilde{\delta_2}^\lambda(w) = (w + (1 - \lambda))^{\theta/\pi}.$$



(a) The set $\tilde{\delta_1}^{\lambda}(\mathcal{D})$, for $\lambda = 0.9$.

(b) For $\lambda = 0.9$ and $\theta = \pi/4$, the boundary of the set $\tilde{\delta}^{\lambda}(\mathcal{D})$ is shown in solid lines and the boundary of $\mathcal{D}^{\theta}$ is shown in dotted lines.

Figure 15: An illustration of the mappings $\tilde{\delta_1}^{\lambda}(\cdot)$ and $\tilde{\delta}^{\lambda}(\cdot)$.

(In the definition of $\tilde{\delta_1}^{\lambda}$ and $\tilde{\delta_2}^{\lambda}$, given $\zeta = \rho e^{\mathrm{j}\varphi}$, with $\rho \geq 0$, and $\varphi \in (-\pi, \pi]$, and given $p \in (0, 1]$, we set $\zeta^p = \rho^p e^{\mathrm{j}p\varphi}$. In other words, the "cut" is taken along the negative real axis.) It is readily checked that, for every $\lambda \in (0, 1)$, $\tilde{\delta}^{\lambda}$ takes real values on the real axis. For fixed $\lambda \in (0, 1)$, $\tilde{\delta}^{\lambda}(\overline{\mathcal{D}})$ is as depicted in Fig. 15(b) (it belongs to $\mathcal{D}^{\theta}$).

As $\lambda \to 1$, the boundary of $\tilde{\delta}^{\lambda}(\overline{\mathcal{D}})$ uniformly approaches that of $\mathcal{D}^{\theta}$. As the next step, we select $\tilde{\delta} = \tilde{\delta}^{\lambda^*}$ where $\lambda^* \in (0, 1)$ is such that both γ and $1/2 + \mathrm{j}0$ belong to the interior of $\tilde{\delta}^{\lambda^*}(\overline{\mathcal{D}})$. We next define $\hat{\delta}$ as a truncated Taylor series of $\tilde{\delta}$ about $1/2 + \mathrm{j}0$, with the properties that $\hat{\delta}(\overline{\mathcal{D}})$belongs to $\mathcal{D}^{\theta}$, and that γ belongs to $\hat{\delta}(\mathcal{D})$. The existence of such $\hat{\delta}$ is a direct consequence of the uniform convergence of the Taylor series. Since $\tilde{\delta}$ is real on the real axis, $\hat{\delta}$ is a polynomial with real coefficients. Further, a real-rational mapping $\check{\delta}$ is defined as the composition of the mapping $s \mapsto (s-1)/(s+1)$, which maps $\overline{\mathbb{C}_+}$ to $\overline{\mathcal{D}}$, with $\xi\hat{\delta}$, where $\xi \in (0, 1)$ is such that γ belongs to the boundary of $\xi\hat{\delta}(\mathcal{D})$. It is readily checked that the image under $\check{\delta}$ of the imaginary axis is this boundary. Also, since $\check{\delta}(\mathbb{C}_+)$ belongs to $\mathcal{D}^{\theta}$, it obviously is bounded in the right half plane. Finally, we let $\delta(s) = \check{\delta}(\frac{\tilde{\omega}}{\tilde{\omega}}s)$, where $\tilde{\omega} \in \mathbb{R} \cup \{\infty\}$ is such that $\check{\delta}(\mathrm{j}\tilde{\omega}) = \gamma$. (In particular, if $\tilde{\omega} \in \{0, \infty\}$, γ is real (since $\check{\delta}$ is real rational) and $\delta(s) = \gamma$ for all s.) Clearly, δ has all the claimed properties. $\square$

Acknowledgement. The authors wish to thank Dr. Carlos Berenstein for his helpful suggestions in connection with Proposition 2, and Dr. Pramod Khargonekar for pointing out the non-obviousness of the necessity part of the small-μ theorem in the present context.

References

[1] J. Doyle. Analysis of feedback systems with structured uncertainties. *IEE Proc.*, 129-D(6):242–250, November 1982.

[2] M. G. Safonov. Stability margins of diagonally perturbed multivariable feedback systems. *IEE Proc.*, 129-D(6):251–256, November 1982.

[3] A. Packard, J. C. Doyle, and G. J. Balas. Linear, multivariable robust control with a μ perspective. *ASME Journal on Dynamics, Measurements and Control, Special Edition on Control*, 115(2b):426–438, June 1993.

[4] W. Levine, editor. *The Control Handbook*. CRC Press, 1995.

[5] A. İftar and Ü. Özgüner. Modeling of uncertain dynamics for robust controller design in state space. *Automatica*, 1991.

[6] W. M. Haddad and D. S. Bernstein. Robust stabilization with positive real uncertainty: Beyond the small gain theorem. *Syst. Control Letters*, 1991.

[7] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. Academic Press, New York, 1975.

[8] A. Packard and J. Doyle. The complex structured singular value. *Automatica*, 29(1):71–109, 1993.

[9] I. Postlethwaite, J. M. Edmunds, and A. G. J. MacFarlane. Principal gains and principal phases in the analysis of linear multivariable feedback systems. *IEEE Trans. Aut. Control*, 1981.

[10] Y. S. Hung and A. G. J. MacFarlane. *Multivariable feedback: a quasi-classical approach*. Springer-Verlag, 1982.

[11] D. H. Owens. The numerical range: A tool for robust stability studies? *Syst. Control Letters*, 1984.

[12] L. Lee, A. L. Tits, and M. K. H. Fan. Robustness under uncertainty with phase information. In *Proc. IEEE Conf. on Decision and Control*, pages 2315–2316, Tampa, Florida, December 1989.

[13] L. Lee and A. L. Tits. Linear fractional transformations for the approximation of various uncertainty sets. In S.P. Bhattacharyya and L. Keel, editors, *Control of Uncertain Systems*, pages 53–62. CRC Press, 1992.

[14] L. Lee and A. L. Tits. On phase information in multivariable systems. In H. Kimura and S. Kodama, editors, *Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing I., Proceedings of the International Symposium MTNS-91—Volume I*, pages 221–226. Mita Press, 1992.

[15] L. Lee. *Robustness Study of Systems with Phase-Informed Uncertainty.* PhD thesis, Department of Electrical Engineering, University of Maryland, 1992.

[16] E.G. Eszter and C.V. Hollot. An IQC for uncertainty satisfying both norm-bounded and passivity constraints. *Automatica*, 33(8):1545–1548, 1997.

[17] A. L. Tits. The small μ theorem without real-rational assumption. In *Proceedings of the 1995 American Control Conference*, pages 2870–2872, 1995.

[18] A. L. Tits and V. Balakrishnan. Small-μ theorems with frequency-dependent uncertainty bounds. 1997. Under review, *MCSS*, 1997.

[19] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. Aut. Control*, 42(6):819–830, June 1997.

[20] J. C. Doyle. Structured uncertainty in control system design. In *Proc. IEEE Conf. on Decision and Control*, Fort Lauderdale, Florida, December 1985.

[21] M. K. H. Fan, A. L. Tits, and J. C. Doyle. Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. *IEEE Trans. Aut. Control*, AC-36(1):25–38, January 1991.

[22] V. Balakrishnan. Linear matrix inequalities in robustness analysis with multipliers. *Syst. Control Letters*, 25(4):265–272, 1995.

[23] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics.* SIAM, Philadelphia, PA, June 1994.

[24] P. Gahinet and A. Nemirovskii. *LMI Control Toolbox: the LMI Lab.* The MathWorks, Inc., 1995.

[25] L. El Ghaoui, F. Delebecque, and R. Nikoukhah. *LMITOOL: A User-friendly Interface for LMI Optimization.* ENSTA/INRIA, 1995. Software available via anonymous FTP from `ftp.inria.fr`, under directory `pub/elghaoui/lmitool`.

[26] C. T. Lawrence, A. L. Tits, and P. Van Dooren. A fast algorithm for the computation of an upper bound on the μ-norm. In *Proceedings of the 13th IFAC World Congress, Volume* H, pages 59–64, 1996.

[27] Y. S. Chou, A. L. Tits, and V. Balakrishnan. On the relationship between stability multipliers and scaling matrices in mixed μ analysis. 1997. Under review, *IEEE Trans. Aut. Control*, 1997.

[28] J. C. Willems. Least squares stationary optimal control and the algebraic Riccati equation. *IEEE Trans. Aut. Control*, AC-16(6):621–634, December 1971.

[29] B. Anderson and S. Vongpanitlerd. *Network analysis and synthesis: a modern systems theory approach.* Prentice-Hall, 1973.

[30] V. Balakrishnan and F. Wang. Efficient computation of a guaranteed lower bound on the robust stability margin of uncertain systems. Submitted to the *IEEE Trans. Aut. Control, 1998.*

[31] L. Vandenberghe and S. Boyd. A primal-dual potential reduction method for problems involving matrix inequalities. *Mathematical Programming*, 69(1):205–236, July 1995.