

# On Kharitonov's Theorem without Invariant Degree Assumption

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## Abstract

The original statement of Kharitonov's theorem requires that all polynomials in the family have the same degree. It has been shown recently that such an assumption is unnecessary. Here we show that the validity of the stronger statement follows from a classical proof combined with additional elementary considerations.

## 1 Introduction

The original statement of Kharitonov's theorem [3] requires that all polynomials in the family have the same degree. It has been shown recently [4, 2] that such an assumption is unnecessary. Here we show that the validity of the stronger statement follows from a classical proof (see, e.g. [1]) combined

with additional elementary considerations.<sup>1</sup>

## 2 Statement and proof

Given  $q_i^-, q_i^+ \in \mathbb{R}$  with  $q_i^- \leq q_i^+$ ,  $i = 0, 1, \dots, n$ , and  $q_n^+$  and  $q_n^-$  not both zero, consider the interval polynomial family

$$P := \{p(s) = q_n s^n + \dots + q_0 \quad : \quad q_i \in [q_i^-, q_i^+], \quad i = 0, 1, \dots, n\}$$

and the four associated ‘‘Kharitonov polynomials’’

$$K_1(s) := q_n^- s^n + q_{n-1}^- s^{n-1} + q_{n-2}^+ s^{n-2} + q_{n-3}^+ s^{n-3} + q_{n-4}^- s^{n-4} + \dots$$

$$K_2(s) := q_n^- s^n + q_{n-1}^+ s^{n-1} + q_{n-2}^+ s^{n-2} + q_{n-3}^- s^{n-3} + q_{n-4}^- s^{n-4} + \dots$$

$$K_3(s) := q_n^+ s^n + q_{n-1}^+ s^{n-1} + q_{n-2}^- s^{n-2} + q_{n-3}^- s^{n-3} + q_{n-4}^+ s^{n-4} + \dots$$

$$K_4(s) := q_n^+ s^n + q_{n-1}^- s^{n-1} + q_{n-2}^- s^{n-2} + q_{n-3}^+ s^{n-3} + q_{n-4}^+ s^{n-4} + \dots$$

**Theorem 1** *All polynomials in  $P$  are Hurwitz stable if and only if  $K_1$ ,  $K_2$ ,  $K_3$  and  $K_4$  are.*

**Proof** Necessity is obvious. We proceed to prove sufficiency. Since  $q_n^+$  and  $q_n^-$  are not both zero, there is no loss of generality in assuming that  $q_n^+ > 0$ . Stability of  $K_4$  implies that  $q_{n-1}^- > 0$  and that of  $K_1$  then implies that  $q_n^- \geq 0$ . Since  $q_{n-1}^+ \geq q_{n-1}^- > 0$ , stability of the four  $K_i$ 's implies that all other coefficients are positive, i.e.,  $q_i^+ \geq q_i^- > 0$ ,  $i = 0, 1, \dots, n-2$ . We now proceed by contradiction. Thus let  $\hat{p} \in P$  be unstable. Denote its coefficients by  $\hat{q}_i$ ,  $i = 0, 1, \dots, n$ . Since  $\hat{q}_n \geq q_n^- \geq 0$ , either  $\hat{q}_n > 0$  or  $\hat{q}_n = q_n^- = 0$ . In the former case, define, say,  $\hat{k} := K_3$ , in the latter case, say,  $\hat{k} := K_1$ . In both cases all polynomials in the line segment

$$E := \{(1-t)\hat{p} + t\hat{k} : t \in [0, 1]\}$$

have the same degree ( $n$  or  $n-1$ ) and  $E$  contains both a stable ( $\hat{k}$ ) and an unstable ( $\hat{p}$ ) polynomials. For  $\omega \in \mathbb{R}$  let  $E(j\omega)$  and  $P(j\omega)$  be the value sets

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<sup>1</sup>Note that a continuity argument invoking the original Kharitonov theorem with  $q_n^- = 0$  changed to  $q_n^- = \epsilon > 0$  (in the notation defined below) would fail to prove Hurwitz stability of the face  $q_n^- = 0$ .

associated with  $E$  and  $P$ . It follows from the Zero Exclusion Condition (see, e.g., Lemma 5.7.9 in [1]) applied to  $E$  that, for some  $\omega^*$ ,

$$0 \in E(j\omega^*) \subseteq P(j\omega^*).$$

Also, since  $q_0^+ \geq q_0^- > 0$ , it is clear that  $0 \notin P(j0)$ . The argument used in the proof given in [1], specifically “case 2” p. 78, now applies.<sup>2</sup> The contradiction revealed in that argument completes the proof.

### 3 Conclusion

An elementary proof was provided for Kharitonov’s theorem without invariant degree assumption.

### References

- [1] B.R. Barmish. *New Tools for Robustness of Linear Systems*. Macmillan, New York, 1994.
- [2] R. Hernandez and S. Dormido. Kharitonov’s theorem extension to interval polynomials which can drop in degree: a Nyquist approach. *IEEE Trans. Automat. Control*, 41:1009–1012, 1996.
- [3] V.L. Kharitonov. Asymptotic stability of an equilibrium position of a family of systems of linear differential equations. *Differential’nye Uraveniya*, 14:1483–1485, 1978.
- [4] J.C. Willems and R. Tempo. The Kharitonov theorem with degree drop. Preprint, Rijksuniversiteit Groningen, Groningen, The Netherlands, 1998.

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<sup>2</sup>The notation used here is essentially the same as that used in [1], with however our  $P(j\omega)$  being  $p(j\omega, Q)$  in [1].