Stability Multipliers and μ Upper Bounds: Connections and Implications for Numerical Verification of Frequency Domain Conditions*

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June 29, 1998

Abstract

The main contribution of the paper is to show the *equivalence* between the following two approaches for obtaining sufficient conditions for the robust stability of systems with structured uncertainties: (i) apply the classical absolute stability theory with multipliers; (ii) use modern μ theory, specifically, the μ upper bound obtained by Fan, Tits and Doyle [IEEE TAC, Vol. 36, 25-38]. In particular, the relationship between the stability multipliers used in absolute stability theory and the scaling matrices used in the cited reference is explicitly characterized. The development hinges on the derivation of certain properties of a parameterized family of complex LMIs (linear matrix inequalities), a result of independent interest. The derivation also suggests a general computational framework for checking the feasibility of a broad class of frequency-dependent conditions, based on which bisection schemes can be devised to reliably compute several quantities of interest for robust control.

1 Introduction

A popular paradigm currently in use for robust control has a nominal finite-dimensional, linear, time-invariant system with the uncertainty Δ in the feedback loop (see Figure 1). Often additional information about the uncertainty is either known or assumed: diagonal or block-diagonal; sector-bounded memoryless, linear time-invariant or parametric, etc. In such cases, the uncertainty is called "structured".

A fundamental question associated with this model is that of robust stability, i.e., "Is the model stable irrespective of the uncertainty Δ, that is, with zero input, do all solutions of the system equations go to zero, irrespective of Δ?" The origins of this question can be found in Russian

*This research was in part supported by NSF's Engineering Research Center No. NSFD-CDR-88-03012, and in part by the Office of Naval Research under contract No. N00014-97-1-0640.

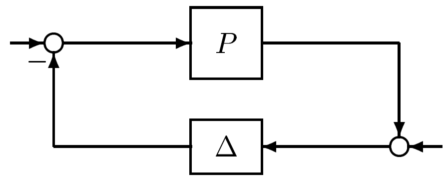

Figure 1: P-Δ loop

literature, where stability of the system in Figure 1 was studied for single-input single-output case, and Δ is required to satisfy additional assumptions. This was known as the *absolute stability problem* [1]. This problem has received considerable attention over the years, and a number of sufficient conditions for stability have been proposed; perhaps the most celebrated of these have been the circle and Popov criteria. These criteria have since been generalized to multi-input multi-output systems as the small-gain theorem (with loop transformations and multipliers) and the passivity theorem (with loop transformations and multipliers). An introduction to these methods can be found in the book by Desoer and Vidyasagar [2]. A second approach to the problem of robust stability of control systems with structured uncertainties is the modern μ (or structured singular value) approach, pioneered by Doyle [3, 4] and Safonov [5, 6]. This approach relies on deriving sufficient conditions for the robust stability of the system in Figure 1 through simple linear-algebraic techniques.

Our main objective in this paper is to show explicitly and rigorously the connections between these two approaches in the case when Δ is linear time-invariant and consists of both unmodeled dynamics and uncertain parameters.[1] In this context, a sufficient condition for robust stability of the loop of Figure 1, derived using the μ approach, is given in [11]; and a sufficient condition based on the classical passivity theorem with multipliers can be found in [10, 12, 13, 14]. The former, which we will refer to as the standard mixed μ upper bound condition, can be recast as an LMI condition that must be satisfied at every boundary point of a "stability" region (the open unit disk or the open left half complex plane) in the complex plane. The latter passivity-multiplier-based robust stability condition in [10, 12, 13, 14] can be reinterpreted as the same LMI condition, with the additional restriction that the feasible solutions are themselves *functions* of a certain form. Therefore, the first step in our treatment is to establish a general "interpolation" style result for a class of parameterized (by frequency) family of complex LMIs (many frequency-dependent conditions for stability and robustness [11, 15, 16, 17, 18, 19] belong to this class). This result is of independent interest.

Thus, we show that the standard mixed μ upper bound condition in [11] is mathematically *equivalent* to the passivity-multiplier-based condition in [10, 12, 13, 14]. Concurrently, we explicitly characterize the relationship between the (D, G) scalings used in the standard mixed μ approach, and certain stability multipliers used in the passivity-multiplier-based condition in [10, 12, 13, 14]. Finally, our derivation also suggests a general computational framework for checking the feasibility

[1]Other authors have hinted at the connection [7], [8], [9], [10], but without proving equivalence.

of a broad class of frequency-dependent conditions.

We first carry out the analysis in the discrete-time context, then briefly indicate how it extends to the continuous-time context. The organization of the paper is as follows. In Section 2, we present our study of a class of LMIs. Section 3 is concerned with the connections between the standard mixed μ upper bound condition and a passivity-multiplier-based robust stability condition. Implications of the results of Sections 2 and 3 on computation are discussed in Section 4. In Section 5, we outline the continuous-time case results. Section 6 contains the conclusions. For clarity of presentation, all proofs have been relegated to the appendix. Preliminary versions of some of the results presented here appeared in [20] and [21].

The notation and terminology are standard. $\mathbf{R}_e$ denotes the set of extended real numbers $\mathbf{R} \cup \{\infty\}$, $\mathbf{C}_-$ the open left-half complex plane $\{z \in \mathbf{C} : \mathrm{Re}\{z\} < 0\}$, and $\mathbf{D}$ the open unit-disk $\{z \in \mathbf{C} : |z| < 1\}$, $\partial\mathbf{D}$ the unit circle, and cl$\mathbf{D}$ the closure of $\mathbf{D}$. $\mathbf{RH}_\infty^{n\times n}$ denotes the set of $n \times n$ real rational proper transfer function matrices with no poles in the complement of $\mathbf{D}$. Given a complex matrix/scalar M, we denote by $\overline{M}$ the complex conjugate of M, by M^* the conjugate transpose of M, by $\mathrm{He}(M)$ the Hermitian part $\frac{1}{2}(M+M^*)$ of M, and by $\mathrm{Sh}(M)$ the skew Hermitian part $\frac{1}{2}(M-M^*)$ of M.

2 Framework: Parametrized Families of Complex LMIs

A linear matrix inequality (LMI) is a matrix inequality of the form

$$L\left(x_1, \cdots, x_\ell, M_0, \cdots, M_\ell\right) := M_0 + \sum_{i=1}^{\ell} x_i M_i + \left(M_0 + \sum_{i=1}^{\ell} x_i M_i\right)^* > 0, \tag{1}$$

where $x \in \mathbf{C}^\ell$ is the variable, and $M_i \in \mathbf{C}^{n\times n}, i = 0, \ldots, \ell$ are given data. The inequality symbol indicates positive definiteness. LMIs (1) are widely encountered in system and control theory and their applications; see, for example [22]. In this section, we consider a parametrized family of LMIs, every element of which is of the form (1). The variable z parametrizing this family takes on values on the unit circle. Using arguments from the theory of complex variables, we show that if every member of this parametrized family of LMIs is feasible, then a solution to the entire family exists in the form of a complex-valued function with certain strong properties. We will see later that such statements help establish the equivalence between two classes of numerical procedures for checking (approximately) the feasibility of the parametrized family of LMIs: The first procedure relies on "sampling" along $\partial\mathbf{D}$ resulting in a number of independent LMIs of the form (1). The second procedure involves searches for a function (of the parametrizing variable z) of a certain form, whose values sampled along $\partial\mathbf{D}$ serve as feasible points for the parametrized family of LMIs; this results in a single LMI of the form (1).

Let $\mathbf{N}$ be the set of nonnegative integers, and let

$$\mathcal{F}_{\mathbf{D}} := \left\{ F : \partial\mathbf{D} \to \mathbf{C}^{n\times n} :\ F \text{ is continuous, with } F(\overline{z}) = \overline{F(z)} \right\},$$

and

$$\mathcal{H}_{\mathrm{FIR}} := \left\{ \sum_{i=0}^{N} \left(a_i z^i + b_i z^{-i} \right) : N \in \mathbf{N}, a_i, b_i \in \mathbf{R} \right\}.$$

The next proposition establishes properties of solutions of the complex parameterized LMI

$$L\left(x_1, \cdots, x_\ell, F_0(z), \cdots, F_\ell(z)\right)) > 0, \tag{2}$$

when $F_i \in \mathcal{F}_{\mathbf{D}}$, $i = 0, \ldots, \ell$.

Proposition 2.1 [2] *Let $F_i \in \mathcal{F}_{\mathbf{D}}$, $i = 0, \ldots, \ell$. Let S be the set of functions s mapping $\mathrm{cl}\mathbf{D}$ to $\mathbf{C}$, which are continuous on $\mathrm{cl}\mathbf{D}$, analytic in $\mathbf{D}$, and satisfy $s(\overline{z}) = \overline{s(z)}$ for all $z \in \mathrm{cl}\mathbf{D}$. The following conditions are equivalent:*

(a) *for every $z \in \partial\mathbf{D}$, there exist complex numbers x_i, $i = 1, \ldots, \ell$, such that* (2) *holds;*

(b) *there exist $s_i^j \in S$, $i = 1, \ldots, \ell$, $j = 1, 2$, such that* (2) *holds for all $z \in \partial\mathbf{D}$ with $x_i(z) = s_i^1(z) + \overline{s_i^2(z)}$, for all $z \in \partial\mathbf{D}$, $i = 1, \ldots, \ell$;*

(c) *there exist $x_i \in \mathcal{H}_{\mathrm{FIR}}$, $i = 1, \ldots, \ell$, such that* (2) *holds for all $z \in \partial\mathbf{D}$.*

A similar result can be proved with $\mathbf{D}$ replaced by an essentially arbitrary complex domains Ω, S defined accordingly, and $\partial\Omega$ replaced by the extended boundary $\partial_{\mathrm{e}}\Omega = \partial\Omega \cup \{\infty\}$ when Ω is unbounded. The appropriate modification of statement (c) is obtained by replacing polynomials in z by linear combinations of functions from an arbitrary countable basis φ_i for S and polynomials in z^{-1} by linear combinations of functions ψ_i such that $\psi_i(z) = \overline{\varphi_i(z)}$ on $\partial_{\mathrm{e}}\Omega$. See [21] for details.

Remark: The equivalence of Statements (a) and (b) in Proposition 2.1—that LMI (2) is feasible for all $z \in \partial\mathbf{D}$ if and only if there exist functions with certain smoothness properties whose values on $\partial\mathbf{D}$ lie in the feasible set—taken together with approximation theory, has important consequences. To wit, it implies that, when LMI (2) arises from the application of multiplier theory in robust stability analysis, the x_i's can be taken as a linear combination of largely arbitrary basis functions, provided the basis is rich enough. See [23] for various possible bases, Laguerre and Kautz bases in particular. As another example, Statement (c) uses an FIR basis.

3 Small μ Theorem and Classical Passivity Theorem

We now apply the results of §2 to explore the connection between two popular sufficient conditions for the robust stability of discrete-time systems with structured uncertainty. The first is the standard mixed μ upper bound condition, given in [11]. This condition is derived using linear-algebraic methods, and is usually stated as an LMI condition that should be satisfied on the unit circle. The derivation of the second condition involves augmenting the system with multipliers (that are

[2]This is an generalization of Lemma 2.3 of [15]. A complete proof is provided for completeness and ease of reference. Parts of this proof, derived independently of the proof in [15], appeared in [20].

introduced to take advantage of the structure and the nature of the uncertainty), and then applying the classical passivity theorem [10, 12, 13, 14]. Proposition 2.1 then helps establish the *equivalence* between these two conditions.

A key tool in establishing the equivalence is a sufficient condition for *canonical factorization*, first stated and proved by A. Ran and his coauthors [24, 25], with the proof building on ideas from [26]. Related results can be found in [27] and [2, §VI.9.4].

Proposition 3.1 *[24, 25] Suppose that M is an $m \times m$ transfer function matrix, not necessarily proper, with no poles on the unit circle, such that the frequency domain condition*

$$M(z) + (M(z)^* > 0 \quad \forall z \in \partial \mathbf{D} \tag{3}$$

holds. Then there exist $M_1, M_2 \in \mathbf{RH}_\infty^{m\times m}$, such that $M_1^{-1}, M_2^{-1} \in \mathbf{RH}_\infty^{m\times m}$, and $M = M_1 M_2^\sim$.

Remark: In addition to the hypotheses of Proposition 3.1, suppose further that $M(z)$ is Hermitian for all $z \in \partial\mathbf{D}$. Then there exists $M_1 \in \mathbf{RH}_\infty^{m\times m}$ such that $M_1^{-1} \in \mathbf{RH}_\infty^{m\times m}$, and $M = M_1 M_1^\sim$. This is the *spectral factorization* of M; it is well-known that condition (3) is both necessary and sufficient for such a factorization (see, for example, [27]). Note however that condition (3) is only sufficient for a canonical factorization of M.

Before proceeding further, we need the following definitions and notation that are standard in μ analysis. Given a matrix $M \in \mathbf{C}^{n\times n}$ and three nonnegative integers m_r, m_c, and m_C with $m := m_\mathrm{r} + m_\mathrm{c} + m_\mathrm{C} \le n$, a block structure $\mathcal{K}$ of dimensions $(m_\mathrm{r}, m_\mathrm{c}, m_\mathrm{C})$ is an m-tuple of positive integers, $\mathcal{K} := (k_1^\mathrm{r}, \ldots, k_{m_\mathrm{r}}^\mathrm{r}, k_1^\mathrm{c}, \ldots, k_{m_\mathrm{c}}^\mathrm{c}, k_1^\mathrm{C}, \ldots, k_{m_\mathrm{C}}^\mathrm{C})$, such that $\sum_{i=1}^{m_\mathrm{r}} k_i^\mathrm{r} + \sum_{i=1}^{m_\mathrm{c}} k_i^\mathrm{c} + \sum_{i=1}^{m_\mathrm{C}} k_i^\mathrm{C} = n$. Let $n_c := \sum_{i=1}^{m_\mathrm{c}} k_i^\mathrm{c}$, and $n_\mathrm{C} := \sum_{i=1}^{m_\mathrm{C}} k_i^\mathrm{C}$. Define

$$\begin{array}{lcl}
\tilde{\mathcal{D}} & := & \left\{ \begin{array}{l} \mathrm{diag}(D_1^\mathrm{r}, \ldots, D_{m_\mathrm{r}}^\mathrm{r}, D_1^\mathrm{c}, \ldots, D_{m_\mathrm{c}}^\mathrm{c}, \\ \qquad d_1^\mathrm{C} I_{k_1^\mathrm{C}}, \ldots, d_{m_\mathrm{C}}^\mathrm{C} I_{k_{m_\mathrm{C}}^\mathrm{C}}) \end{array} : \begin{array}{l} D_i^\mathrm{r} = (D_i^\mathrm{r})^* \in \mathbf{C}^{k_i^\mathrm{r}\times k_i^\mathrm{r}}, \\ D_i^\mathrm{c} = (D_i^\mathrm{c})^* \in \mathbf{C}^{k_i^\mathrm{c}\times k_i^\mathrm{c}}, d_i^\mathrm{C} \in \mathbf{R} \end{array} \right\} \\
\mathcal{D} & := & \left\{ D \in \tilde{\mathcal{D}} : D > 0 \right\} \\
\mathcal{G} & := & \left\{ \mathrm{diag}(G_1^\mathrm{r}, \ldots, G_{m_\mathrm{r}}^\mathrm{r}, 0_{k_1^\mathrm{c}}, \ldots, 0_{k_{m_\mathrm{c}}^\mathrm{c}}, 0_{k_1^\mathrm{C}}, \ldots, 0_{k_{m_\mathrm{C}}^\mathrm{C}}) : G_i^\mathrm{r} = -(G_i^\mathrm{r})^* \in \mathbf{C}^{k_i^\mathrm{r}\times k_i^\mathrm{r}} \right\} \\
\hat{\mu}(M) & := & \inf_{\beta\ge 0, D\in\mathcal{D}, G\in\mathcal{G}} \{\beta : MDM^* + GM^* - MG - \beta^2 D < 0\}.
\end{array}$$

Next, let $\mathbf{RF}$ denote the set of real-rational functions and, given $X = \mathbf{C}$ or $\mathbf{RF}$, define

$$\begin{array}{lcl}
\mathcal{S}_\mathrm{r}(X) & := & \left\{ \mathrm{diag}(S_1, \ldots, S_{m_\mathrm{r}}) : S_i \in X^{k_i^\mathrm{r}\times k_i^\mathrm{r}}, i = 1, \ldots, m_\mathrm{r} \right\} \\
\mathcal{S}_\mathrm{c}(X) & := & \left\{ \mathrm{diag}(S_1, \ldots, S_{m_\mathrm{c}}) : S_i \in X^{k_i^\mathrm{c}\times k_i^\mathrm{c}}, i = 1, \ldots, m_\mathrm{c} \right\} \\
\mathcal{S}_\mathrm{C}(X) & := & \left\{ \mathrm{diag}(s_1 I_{k_1^\mathrm{C}}, \ldots, s_{m_\mathrm{C}} I_{k_{m_\mathrm{C}}^\mathrm{C}}) : s_i \in X, i = 1, \ldots, m_\mathrm{C} \right\} \\
\mathcal{S}_\mathrm{c,C}(X) & := & \{\mathrm{diag}(S_\mathrm{c}, S_\mathrm{C}) : S_\mathrm{c} \in \mathcal{S}_\mathrm{c}(X), S_\mathrm{C} \in \mathcal{S}_\mathrm{C}(X)\} \\
\mathcal{S}_\mathrm{r,c,C}(X) & := & \{\mathrm{diag}(S_\mathrm{r}, S_\mathrm{c}, S_\mathrm{C}) : S_\mathrm{r} \in \mathcal{S}_\mathrm{r}(X), S_\mathrm{c} \in \mathcal{S}_\mathrm{c}(X), S_\mathrm{C} \in \mathcal{S}_\mathrm{C}(X)\} \\
\mathcal{S}_\mathrm{r,0,0}(X) & := & \{\mathrm{diag}(S_\mathrm{r}, 0_{n_\mathrm{c}}, 0_{n_\mathrm{C}}) : S_\mathrm{r} \in \mathcal{S}_\mathrm{r}(X)\}
\end{array}$$

Finally, given a matrix/scalar function P continuous on $\partial\mathbf{D}$, define

$$\begin{array}{l}
\|P\|_{\hat{\mu}} := \sup_{z\in\partial\mathbf{D}} \hat{\mu}(P(z)) \qquad \text{(note that it is not a norm)} \\
P^\sim \text{ is defined by } P^\sim(z) := (P(1/\bar{z}))^* \\
\tilde{P} := (I + P)(I - P)^{-1}
\end{array}$$

We now state Theorem 3.2, which essentially establishes that the standard mixed μ upper bound condition is mathematically equivalent to a passivity-multiplier-based stability condition: strict positive realness of $W_2^{-1}\tilde{P}W_1^{-1}$ with "multipliers" W_1 and W_2 belonging to a certain subset of $\mathbf{RH}_\infty^{n\times n}$ (see Fig. 2 where $\tilde{\Delta} := (I+\Delta)(I-\Delta)^{-1}$).[3] In addition, the relationship between

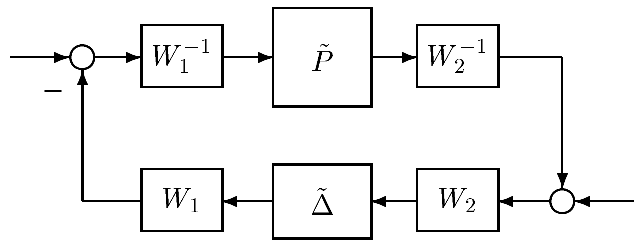

Figure 2: Passivity framework

(W_1, W_2) and the (D, G) scalings used in the mixed μ analysis is explicitly characterized.

Theorem 3.2 [4]

Let $P \in \mathbf{RH}_\infty^{n\times n}$. The following conditions are equivalent:

(a)

$$\|P\|_{\hat{\mu}} < 1,$$

(b) $(I-P)^{-1}$ *is in* $\mathbf{RH}_\infty^{n\times n}$ *and there exist a positive integer* N, *and* $n\times n$, *real matrices* $Q_i \in \mathcal{S}_{\mathrm{r,c,C}}(\mathbf{C})$, *and* $U_i \in \mathcal{S}_{\mathrm{r,0,0}}(\mathbf{C})$, $i = 1, \ldots, N$, *such that with*

$$D(z) = \sum_{i=0}^{N}(Q_i z^i + Q_i^T z^{-i}), \tag{4}$$

$$G(z) = \sum_{i=0}^{N}(U_i z^i - U_i^T z^{-i}), \tag{5}$$

the inequalities

$$D(z) > 0, \tag{6}$$

$$P(z)D(z)(P(z))^* + G(z)(P(z))^* - P(z)G(z) - D(z) < 0 \tag{7}$$

hold for all $z \in \partial\mathbf{D}$,

[3]Recall that a rational transfer matrix T is *strictly positive real* if it is in $\mathbf{RH}_\infty^{n\times n}$ and $\mathrm{He}(T(z)) > 0$ for all $z \in \partial\mathbf{D}$.

[4]Many researchers have been aware of this result for some time (see, e.g., [28]). (Our proof already appears in [21]; a less general version can be found in [20].) To our knowledge however, equivalence between (a) and any of the other conditions has never been rigorously established by other authors in the open literature, nor has been the continuous-time case counterpart. Implication (c)$\Rightarrow$(b), which is of little importance to us and, in our proof, follows from other implications, is immediate from fixed order multiplier results [29]. Implication (c)$\Rightarrow$(d) is proved in [28] (which also includes a proof of equivalence between (b), (c), and (d) in the constant matrix case). The correspondence between (D, G) scalings and passivity multipliers (last statement in our theorem and an immediate consequence of the equivalence between (a), (b), (c), and (d)) has been alluded to by several authors, e.g., [29]. Finally, while the implication (d)$\Rightarrow$(c) would follow, in the continuous-time case, from results in [12], a key result there is incorrect, namely, the necessary part of Lemma 6: $M(s) = \mathrm{diag}(1, s^4+1)$ provides a simple counterexample.

(c) $(I-P)^{-1}$ *is in* $\mathbf{RH}_\infty^{n\times n}$ *and there exist a positive integer* N, *and* $n\times n$, *real matrices* $R_i, S_i \in \mathcal{S}_{\mathrm{r,c,C}}(\mathbf{C})$, $i=1,\dots,N$, *such that with*

$$W(z)=\sum_{i=0}^{N}(R_i z^i + S_i z^{-i}), \tag{8}$$

the lower right $(n_{\mathrm{c}}+n_{\mathrm{C}})\times(n_{\mathrm{c}}+n_{\mathrm{C}})$ *submatrix* $W_{\mathrm{cC}}(z)$ *of* $W(z)$ *is Hermitian for all* $z\in\partial\mathbf{D}$, *and the inequalities*

$$\begin{aligned}&\mathrm{He}(W(z))>0,\\ &\mathrm{He}(\tilde{P}(z)W(z))>0,\end{aligned}$$

hold for all $z\in\partial\mathbf{D}$,

(d) $(I-P)^{-1}$ *is in* $\mathbf{RH}_\infty^{n\times n}$ *and there exist transfer function matrices* $W_1 = diag(W_1^{\mathrm{r}}, W_1^{\mathrm{c}}, W_1^{\mathrm{C}})$ *and* $W_2 = diag(W_2^{\mathrm{r}}, W_2^{\mathrm{c}}, W_2^{\mathrm{C}})$, *where* W_1^{r} *and* W_2^{r}, W_1^{c} *and* W_2^{c}, *and* W_1^{C} *and* W_2^{C} *are in* $\mathcal{S}_{\mathrm{r}}(\mathbf{RF})$, $\mathcal{S}_{\mathrm{c}}(\mathbf{RF})$, *and* $\mathcal{S}_{\mathrm{C}}(\mathbf{RF})$, *respectively, satisfying*

(i) W_1, W_2, W_1^{-1}, *and* W_2^{-1} *are in* $\mathbf{RH}_\infty^{n\times n}$, $W_1^{\mathrm{c}}W_2^{\mathrm{c}}=I$, *and* $W_1^{\mathrm{C}}W_2^{\mathrm{C}}=I$,

(ii) W_1W_2 *is strictly positive real,*

(iii) $W_2^{-1}\tilde{P}W_1^{-1}$ *is strictly positive real.*

Next, let D *and* G *be as in* (b) *and let* $W=D+G$; *then* W *is as in* (c). *Also, if* W *is as in* (c) *then* $W=W_1^{-1}W_2^{\sim}$ *for some* W_1 *and* W_2 *which are as in* (d). *Finally, let* W_1 *and* W_2 *be as in* (d) *and let* $D=\mathrm{He}(W_1^{-1}W_2^{\sim})$ *and* $G=\mathrm{Sh}(W_1^{-1}W_2^{\sim})$; *then* D *and* G *satisfy conditions (6) and (7) in* (b).

In addition to establishing the precise equivalence between modern mixed μ-theory based sufficient condition for robust stability and the classical passivity-multiplier-based condition, Theorem 3.2 also states that the mutlipliers can be chosen to be of a particularly simple form, given in (4) and in (5). This fact has important ramifications for the numerical verification of these robust stability conditions; we describe this briefly next.

4 State-Space Verification of Frequency Domain Conditions

Many frequency-domain conditions for stability and robustness [11, 15, 16, 17, 18, 19] belong to the class of parameterized families of complex LMIs that were studied in Section 2. From the viewpoint of numerical optimization, these conditions amount to infinite-dimensional convex feasibility problems, with the optimization variables being functions of frequency. There have been traditionally two approaches towards (approximately) reducing such an infinite-dimensional feasibility problem to the feasibility of a finite number of LMIs. The first is to verify the frequency-dependent condition approximately by checking that it holds at a finite number of frequencies; the choice of the "frequency-grid" is typically based on engineering intuition. This frequency-gridding technique, though conceptually simple, in general only provides guaranteed method for verifying the infeasibility of the underlying infinite-dimensional feasibility problem: If the feasibility test fails at any of the points on the frequency-grid, then the original frequency-domain condition is infeasible; but no conclusions can be drawn in the case when feasibility holds over the frequency-grid (for an exception, see, e.g., [30]).

The second approach towards numerically verifying the infinite-dimensional convex feasibility problems that arise from frequency-domain conditions is a "state-space" approach, based on the Kalman-Yakubovich-Popov (KYP) Lemma (see for example [22] and the references therein). The first step here is to restrict the functions of frequency that are the optimization variables in the original frequency-domain condition to lie in a finite-dimensional subspace. Then, an application of the KYP Lemma yields a single LMI whose feasibility is sufficient for that of the original frequency-domain condition. (This is the so-called "basis function method" [12, 13, 14].) Here, it is not known if and how the choice of basis functions affects the outcome of the stability test. In contrast with the frequency-gridding technique, the basis function method provides guaranteed method for verifying the feasibility of the underlying infinite-dimensional feasibility problem: If the frequency-domain condition is feasible with the optimization variables restricted to lie in a certain subspace, then the original frequency-domain condition is feasible; but in general no conclusions can be drawn in the case when the basis function method fails to establish feasibility.

A fundamental question that then arises is whether there is a "gap" between the basis function method and the frequency-gridding method. The main implication of the results of Section 2 is that there is no such "gap": *When the grid is fine enough with the frequency-gridding method, and when the basis is rich enough with the basis function method, the two numerical methods yields the "same" answer.* Specifically, Proposition 2.1 and the comments that follow imply that the frequency-gridding method and the basis function method are *equivalent* in the limit, as the number of frequency points and the basis elements respectively in each method goes to infinity. Consequently, algorithms can be devised that reliably compute quantities of interest for robust control; For example, consider the mixed-μ-norm upper bound $\|P\|_{\hat{\mu}}$. Lower bounds on $\|P\|_{\hat{\mu}}$ can be computed using frequency-gridding (for example, as with the (real-) μ toolbox of MATLAB [4]); upper bounds can be computed using the basis-function method, using Theorem 3.2. A simple bisection scheme can then be implemented, combining these bounds, to compute $\|P\|_{\hat{\mu}}$ to any arbitrary accuracy.

5 Synopsis of the Continuous-Time Case

Continuous-time analogues of Proposition 2.1 and Theorem 3.2 are easily obtained. Among other possibilities, $\mathcal{H}_{\text{FIR}}$ can be replaced with $\mathcal{H}_{\text{UDR}}$ defined by

$$\mathcal{H}_{\text{UDR}} := \left\{ \sum_{i=0}^{N} \left(\frac{a_i}{(1-z)^i} + \frac{b_i}{(1+z)^i} \right) : N \in \mathbf{N},\ a_i, b_i \in \mathbf{R} \right\}.$$

(The subscript "UDR" stands for unit-decay rate; the two-sided inverse Laplace transforms of the elements of $\mathcal{H}_{\text{UDR}}$ decay exponentially with a unit rate of decay.) Proposition 2.1 then holds with $\mathcal{F}_{\mathbf{D}}$ replaced with $\mathcal{F}_{\mathbf{C}_-}$, defined as

$$\mathcal{F}_{\mathbf{C}_-} := \left\{ F : \mathrm{j}\mathbf{R}_{\mathrm{e}} \to \mathbf{C}^{n\times n} :\ F \text{ is continuous, with } F(\overline{z}) = \overline{F(z)} \right\},$$

cl$\mathbf{D}$ by the closure $\mathrm{cl_e}\mathbf{C}_- = \mathrm{cl}\mathbf{C}_- \cup \{\infty\}$ of the extended left-half plane, $\partial\mathbf{D}$ by the extended imaginary axis $\mathbf{j}\mathbf{R}_\mathrm{e}$, and $\mathcal{H}_{\mathrm{FIR}}$ by $\mathcal{H}_{\mathrm{UDR}}$. And Theorem 3.2 holds with, in addition, the appropriate definition of $\|P\|_{\hat{\mu}}$, z^i replaced by $1/(1-z)^i$, z^{-i} by $1/(1+z)^i$ and other obvious modifications. These results are easily proved by making use of the bilinear transformation $\phi(z) = \frac{1+z}{1-z}$ that maps cl$\mathbf{D}$ to the one-point compactification $\mathrm{cl_e}\mathbf{C}_-$ of the closed left-half complex plane.

6 Conclusions

It has been shown that the standard mixed μ upper bound condition is mathematically equivalent to a passivity-multiplier-based stability condition in [10, 12, 13, 14]. Concurrently, the relationship between the scaling matrices widely used in mixed μ theory and certain stability multipliers used in absolute stability theory has been explicitly characterized. These connections provide a computational framework for checking the feasibility of a class of frequency-dependent conditions of interest in robust control (e.g., [11, 15, 16, 17, 18, 19]).

A Appendix

A.1 Proof of Proposition 2.1

Before proving Proposition 2.1, we introduce three lemmas which will be used later.

The first lemma, Lemma A.1, extends a result of Packard and Doyle (Theorem 9.10 in [31]), which explores an intrinsic solution property for a class of parameterized family of complex LMIs.

Lemma A.1 *Let M_i, $i = 0, \ldots, \ell$, be given complex matrices. Suppose that there exist complex numbers x_i, $i = 1, \ldots, \ell$, such that $L(x_1, \cdots, x_\ell, M_0, \cdots, M_\ell) > 0$. Then the inequality also holds with M_i and x_i, $i = 1, \ldots, \ell$ replaced with $\overline{M_i}$ and $\overline{x_i}$, respectively. Furthermore, when the M_i's are real, the inequality holds with x_i, $i = 1, \ldots, \ell$, replaced with $\frac{1}{2}(x_i + \overline{x_i})$, $i = 1, \ldots, \ell$.*

Proof: Recall that, given a Hermitian matrix A, $A > 0$ if and only if $y^*Ay > 0$ for all $y \neq 0$. Taking complex conjugates on both sides of the resulting inequality proves the first claim. The second claim is a direct consequence of the first one. □

Lemma A.2 below is a key to prove Proposition 2.1. With a direct application of the Dirichlet problem on cl$\mathbf{D}$ we are able to establish the lemma, which implies the existence of a combination of functions of certain type that matches on $\partial\mathbf{D}$ a given complex-valued function x satisfying certain properties.

Lemma A.2 *Let $x : \partial\mathbf{D} \to \mathbf{C}$ be a continuous function satisfying $x(\overline{z}) = \overline{x(z)}$ for all $z \in \partial\mathbf{D}$. Then there exist $s^i : \mathrm{cl}\mathbf{D} \to \mathbf{C}$, i=1,2, such that, for i=1,2, s^i is continuous on cl$\mathbf{D}$, analytic in $\mathbf{D}$, and satisfies $s^i(\overline{z}) = \overline{s^i(z)}$ for all $z \in \mathrm{cl}\mathbf{D}$, and $x(z) = s^1(z) + \overline{s^2(z)}$ for all $z \in \partial\mathbf{D}$.*

Proof:[5] Let $x : \partial\mathbf{D} \to \mathbf{C}$ be given as assumed. Let Re$\{x\}$ and Im$\{x\}$ denote its real part and imaginary part, respectively. Then both Re$\{x\}$ and Im$\{x\}$ are real-valued, continuous functions

[5] An alternative proof to that provided here can be found in [20] where explicit (Poisson) formulae for s_1 and s_2 are given. The advantages of the present proof are its conceptual simplicity and the fact that it readily extends to other domains of the complex plane.

defined on $\partial\mathbf{D}$. Consider the Dirichlet problem on $\mathbf{D}$ with the real-valued, continuous boundary function $\mathrm{Re}\{x\}$. It is known that, given a simply connected set X and a real-valued continuous function g defined on ∂X, there exists a function u which is harmonic inside X and matches g on ∂X (see, e.g., Corollary 4.18, pp.274, in [32]). Moreover, for any function u which is harmonic in a simply connected set X, there exists a harmonic conjugate v such that $u + \mathrm{j}v$ is analytic in X (see, e.g., Theorem 2.2(j), pp.202, in [32]). Thus there exists a function analytic in $\mathbf{D}$, whose real part matches $\mathrm{Re}\{x\}$ on $\partial\mathbf{D}$. Define $f'(z) = \overline{f(\overline{z})}$ for all $z \in \mathrm{cl}\mathbf{D}$. It is easy to verify that f' is analytic in $\mathbf{D}$ and its real part matches $\mathrm{Re}\{x\}$ on $\partial\mathbf{D}$. Let $s_1 = \frac{1}{2}(f + f')$. Then s_1 is continuous on $\mathrm{cl}\mathbf{D}$, analytic in $\mathbf{D}$, with $s_1(\overline{z}) = \overline{s_1(z)}$ for all $z \in \mathrm{cl}\mathbf{D}$, and $\mathrm{Re}\{s_1\}(z) = \mathrm{Re}\{x\}(z)$ for all $z \in \partial\mathbf{D}$. Similarly, there exists a function s_2, continuous on $\mathrm{cl}\mathbf{D}$, analytic in $\mathbf{D}$, $s_2(\overline{z}) = \overline{s_2(z)}$ for all $z \in \mathrm{cl}\mathbf{D}$, and $\mathrm{Im}\{s_2\}(z) = \mathrm{Im}\{x\}(z)$ for all $z \in \partial\mathbf{D}$. Let $s^1 = \frac{1}{2}(s_1 + s_2)$ and $s^2 = \frac{1}{2}(s_1 - s_2)$. It follows that, for all $z \in \partial\mathbf{D}$,

$$\begin{aligned} x(z) &= \mathrm{Re}\{x\}(z) + \mathrm{j}\mathrm{Im}\{x\}(z) \\ &= \tfrac{1}{2}(s_1(z) + \overline{s_1(z)}) + \tfrac{1}{2}(s_2(z) - \overline{s_2(z)}) \\ &= \tfrac{1}{2}(s_1(z) + s_2(z) + \overline{s_1(z) - s_2(z)}) \\ &= s^1(z) + \overline{s^2(z)}. \end{aligned}$$

□

Lemma A.3 below, a slight extension of Mergelyan's Theorem (e.g., Theorem 20.5 in [33]), gives conditions for uniform approximation, on a bounded set, of a class of complex-valued functions by polynomials with real coefficients.

Lemma A.3 *Suppose that $s : \mathrm{cl}\mathbf{D} \to \mathbf{C}$ is continuous on $\mathrm{cl}\mathbf{D}$ and analytic in $\mathbf{D}$. Assume that for all $z \in$, $s(\overline{z}) = \overline{s(z)}$. Then given $\epsilon > 0$, there exists a polynomial p with real coefficients, such that*

$$\sup_{z \in \mathrm{cl}\mathbf{D}} \mid s(z) - p(z) \mid < \epsilon.$$

Proof: Given $\epsilon > 0$, by Mergelyan's theorem, there exists a polynomial $p_0 = p + \mathrm{j}q$ where p and q are real-coefficient polynomials, such that

$$\sup_{z \in \mathrm{cl}\mathbf{D}} \mid s(z) - p_0(z) \mid < \epsilon/2. \tag{9}$$

We show that p is as claimed. First, since for all $z \in \mathrm{cl}\mathbf{D}$, $s(\overline{z}) = \overline{s(z)}$, it follows from (9) that, for all $z \in \mathrm{cl}\mathbf{D}$,

$$\begin{aligned} \mid \overline{p_0(z)} - p_0(\overline{z}) \mid &= \mid \overline{p_0(z)} - s(\overline{z}) + s(\overline{z}) - p_0(\overline{z}) \mid \\ &\leq \mid \overline{p_0(z)} - \overline{s(z)} \mid + \mid s(\overline{z}) - p_0(\overline{z}) \mid \\ &< \epsilon. \end{aligned}$$

On the other hand, for all $z \in \mathrm{cl}\mathbf{D}$,

$$\mid \overline{p_0(z)} - p_0(\overline{z}) \mid \;=\; \mid \overline{p(z)} - \mathrm{j}\overline{q(z)} - (p(\overline{z}) + \mathrm{j}q(\overline{z})) \mid .$$

Since, for any real-coefficient polynomial $m(z)$, $m(\overline{z}) = \overline{m(z)}$ for all z, it follows that

$$\mid \overline{p_0(z)} - p_0(\overline{z}) \mid \;=\; 2 \mid q(z) \mid .$$

Thus, for all $z \in \mathrm{cl}\mathbf{D}$, $\mid q(z) \mid < \epsilon/2$. Again using (9), it follows that, for all $z \in \mathrm{cl}\mathbf{D}$,

$$\begin{aligned} \mid s(z) - p(z) \mid &= \mid s(z) - (p(z) + \mathrm{j}q(z)) + \mathrm{j}q(z) \mid \\ &\leq \mid s(z) - p_0(z) \mid + \mid q(z) \mid \; < \; \epsilon. \end{aligned}$$

This proves the claim. □

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1: We show that (a)⇒ (b)⇒(c)⇒ (a).

First the implication (a)⇒(b). Assuming that (a) holds, we show that there exist continuous functions $x_i : \partial\mathbf{D} \to \mathbf{C}$, $i = 1, \ldots, \ell$, such that, for all $z \in \partial\mathbf{D}$, $x_i(\overline{z}) = \overline{x_i(z)}$, $i = 1, \ldots, \ell$, and (2) holds; assertion (b) will then follow from Lemma A.2. The problem of constructing a continuous function on $\partial\mathbf{D}$ based on data x_i's reduces to that of construction a continuous function on the closed interval $[-\pi, \pi]$ with same value assigned at the endpoint $-\pi$ and π. The following argument is similar to that in [15].

For each $\theta \in (-\pi, \pi]$, let x_i^θ, $i = 1, \ldots, \ell$, satisfy

$$L(x_1^\theta, \cdots, x_\ell^\theta, F_0(\mathrm{e}^{\mathrm{j}\theta}), \cdots, F_\ell(\mathrm{e}^{\mathrm{j}\theta})) > 0$$

and let $x_i^{-\pi} = x_i^\pi$, $i = 1, \ldots, \ell$. By Lemma A.1, the x_i^θ's can be chosen such that, for all $\theta \in [0, \pi]$, $i = 1, \ldots, \ell$, $x_i^{-\theta} = \overline{x_i^\theta}$ and, in particular x_i^0 and x_i^π are real. Since the F_i's are continuous, for every $\theta \in [0, \pi]$ there exists an open neighborhood $I^\theta \subset \mathbf{R}$ of θ such that

$$L(x_1^\theta, \cdots, x_\ell^\theta, F_0(\mathrm{e}^{\mathrm{j}\theta'}), \cdots, F_\ell(\mathrm{e}^{\mathrm{j}\theta'})) > 0 \quad \forall \theta' \in I^\theta.$$

We now show that there exists a finite open cover $\{I_1, \cdots, I_m\} \subset \{I^\theta, \theta \in [0, \pi]\}$ with $I_1 = I^0$ and $I_m = I^\pi$, where for $i = 1, \ldots, m$, $I_i = (\alpha_i, \alpha_i')$, $\alpha_1 < 0 < \alpha_2 < \cdots < \alpha_m < \alpha_{m-1}' < \pi < \alpha_m'$, and each I_i is not a subset of the union of the other I_j's. Without loss of generality, assume that $[0, \pi] \setminus (I^0 \cup I^\pi)$ is not empty. Let $\{I_1, \cdots, I_k\} \subset \{I^\theta, \theta \in [0, \pi]\}$ be a finite open cover of the compact interval $[0, \pi] \setminus (I^0 \cup I^\pi)$, and replace each I_i by its intersection with $(0, \pi)$. For $i = 1, \ldots, \ell$, let x_i^j, $j = 1, \ldots, k$, be the associated x_i^θ's. Let $I_0 := I^0$, $I_{k+1} := I^\pi$ and, for $i = 1, \ldots, \ell$, $x_i^{k+1} := x_i^\pi$. Then $\{I_0, \cdots, I_{k+1}\}$ is a finite open cover of the compact interval $[0, \pi]$. Repeatedly take out any interval I_i which is a subset of the union of the remaining I_j's (clearly I^0 and I^π will not be taken out in this process) and denote by m the number of remaining intervals. Finally, relabel the I_i's so that $I_i = (\alpha_i, \alpha_i')$ with α_i's, $i = 1, \ldots, m$ being in increasing order and, for $i = 1, \ldots, \ell$, relabel the x_i^j's accordingly. It is readily checked that the finite cover obtained possesses the desired properties.

Now, for each $i = 1, \ldots, m$, let $V_i := (x_1^i, \cdots, x_\ell^i)$. Then V_1 and V_m are real. Define, for $\theta \in [0, \pi]$,

$$V(\theta) := \begin{cases} V_i & \theta \in I_i - \cup_{j \neq i} I_j, i = 1, \ldots, m. \\ \lambda V_i + (1 - \lambda) V_{i+1} & \theta = \lambda \alpha_{i+1} + (1 - \lambda) \alpha_i', \lambda \in (0, 1), i = 1, \ldots, m - 1. \end{cases}$$

Then $V : [0, \pi] \to \mathbf{C}^\ell$ is continuous. Note that, since $\alpha_2 > 0$ and $\alpha_{m-1}' < \pi$, $V(0)$ and $V(\pi)$ are real. Extending $V(.)$ by the formula $V(-\theta) = \overline{V(\theta)}$ to the interval $[-\pi, 0]$ results in a continuous function defined on $[-\pi, \pi]$ with the same value $V(\pi)$ assigned at the endpoints $-\pi$ and π. This completes the proof of the implication (a)⇒(b).

To show the implication (b)$\Rightarrow$(c), it suffices to show that given $s \in S$ and any $\epsilon > 0$, there always exists a polynomial p with real coefficients which uniformly approximates the given function s over cl$\mathbf{D}$ within ϵ. This result follows immediately from Lemma A.3.

The implication (c)$\Rightarrow$(a) is obvious. This completes the proof of Proposition 2.1.

A.2 Proof of Theorem 3.2

Proof of Theorem 3.2

(a)$\Rightarrow$(b): Suppose that $\|P\|_{\hat{\mu}} < 1$, i.e., suppose that, for every $z \in \partial\mathbf{D}$ there exist $D \in \tilde{\mathcal{D}}$ and $G \in \mathcal{G}$ such that

$$D > 0,$$
$$P(z)D(P(z))^* + G(P(z))^* - P(z)G - D < 0.$$

Since this condition is stronger than the mixed μ condition, it follows from the Small μ Theorem (e.g., [34]) that $(I-P)^{-1} \in \mathbf{RH}_\infty^{n\times n}$. By Proposition 2.1, there exist $\hat{D}(z)$ and $\hat{G}(z)$ whose entries are in $\mathcal{H}_{\text{FIR}}$, such that the inequalities

$$\hat{D}(z) > 0,$$
$$P(z)\hat{D}(z)(P(z))^* + \hat{G}(z)(P(z))^* - P(z)\hat{G}(z) - \hat{D}(z) < 0,$$

hold for all $z \in \partial\mathbf{D}$. Let $D = \hat{D} + \hat{D}^{\sim}$ and $G = \hat{G} - \hat{G}^{\sim}$. Since $\overline{z} = z^{-1}$ for all $z \in \partial\mathbf{D}$, it is easy to verify that $D(z)$ and $G(z)$ have the forms as described and satisfy (6) and (7).

(b)$\Rightarrow$(c): Let $D(z)$ and $G(z)$ be as in (b). Then $D(z) \in \mathcal{D}$, and $G(z) \in \mathcal{G}$ for all $z \in \partial\mathbf{D}$. Define $W(z) = D(z) + G(z) = \sum_{i=0}^{N}((Q_i + U_i)z^i + (Q_i^T - U_i^T)z^{-i})$ with real matrices $Q_i \in \mathcal{S}_{\text{r,c,C}}(\mathbf{C})$, and $U_i \in \mathcal{S}_{\text{r,0,0}}(\mathbf{C})$. Clearly the submatrix $W_{\text{cC}}(z)$ of $W(z)$ is in $\mathcal{S}_{\text{c,C}}(\mathbf{C})$ and is Hermitian for all $z \in \partial\mathbf{D}$. Moreover, since for all $z \in \partial\mathbf{D}$, $D(z)$ is Hermitian and positive definite, and $G(z)$ is skew Hermitian, we have for all $z \in \partial\mathbf{D}$,

$$\text{He}(W(z)) = D(z) > 0.$$

On the other hand, replacing D and G by $\frac{1}{2}(W + W^*)$ and $\frac{1}{2}(W - W^*)$, respectively, in the second inequality in (b) yields, for all $z \in \partial\mathbf{D}$,

$$\begin{aligned}&P(z)(W(z) + (W(z))^*)(P(z))^* + (W(z) - (W(z))^*)(P(z))^* - P(z)(W(z) - (W(z))^*)\\ &-(W(z) + (W(z))^*) < 0,\end{aligned}$$

i.e.,

$$(I + P(z))W(z)(I - P(z))^* + (I - P(z))(W(z))^*(I + P(z))^* > 0. \tag{10}$$

Note that (10) implies that $I - P(z)$ must be nonsingular for all $z \in \partial\mathbf{D}$. Thus it is equivalent to

$$(I - P(z))^{-1}(I + P(z))W(z) + (W(z))^*(I + P(z))^*(I - P(z))^{-*} > 0,$$

i.e.,

$$\tilde{P}(z)W(z) + (\tilde{P}(z)W(z))^* > 0,$$

i.e.,

$$\mathrm{He}(\tilde{P}(z)W(z)) > 0.$$

This proves the claim.

(c)⇒(d): Suppose that (c) holds. Write W as $\mathrm{diag}(W^{\mathrm{r}}, W^{\mathrm{c}}, W^{\mathrm{C}})$, where W^{r}, W^{c}, and W^{C} have dimensions $n_{\mathrm{r}} \times n_{\mathrm{r}}$, $n_{\mathrm{c}} \times n_{\mathrm{c}}$, and $n_{\mathrm{C}} \times n_{\mathrm{C}}$, respectively. Then, $W_{\mathrm{cC}} = \mathrm{diag}\left(W^{\mathrm{c}}, W^{\mathrm{C}}\right)$ is Hermitian for all $z \in \partial\mathbf{D}$. Now note that, under the assumption that (d)*(i)* holds, strict positive realness of $W_1 W_2$ is equivalent to the condition

$$\mathrm{He}\left((W_1(z))^{-1}(W_2(z))^*\right) > 0 \quad \forall z \in \partial\mathbf{D}$$

and strict positive realness of $W_2^{-1}\tilde{P}W_1^{-1}$ is equivalent to the condition

$$\mathrm{He}\left(\tilde{P}(z)(W_1(z))^{-1}(W_2(z))^*\right) > 0 \quad \forall z \in \partial\mathbf{D}$$

(both of these conditions are obtained via congruence transformations). To complete the proof of the implication (c)⇒(d) it is thus enough to factorize W into $W = W_1^{-1}W_2^{\sim}$ where W_1 and W_2 satisfy (d)*(i)*. The desired factorization of W is possible, in view of Proposition 3.1 and the well-known symmetric factorization result mentioned in the remark following that proposition. It remains to apply this result to the block entries of W^{r}, W^{c}, and W^{C} (where W^{r}, W^{c}, and W^{C} all satisfy (3), and both $W^{\mathrm{c}}(z)$ and $W^{\mathrm{C}}(z)$ are Hermitian for all $z \in \partial\mathbf{D}$), and define W_1 and W_2 in the obvious way. This completes the proof of the implication (c)⇒(d).

(d) ⇒ (a): Suppose that (d) holds and let W_1, W_2 satisfy *(i)-(iii)*. In particular

$$W_2^{-1}(z)\tilde{P}(z)W_1^{-1}(z) + (W_1^{-1}(z))^*(\tilde{P}(z))^*(W_2^{-1}(z))^* > 0 \quad \forall z \in \partial\mathbf{D},$$

or, equivalently (via congruence transformation),

$$\tilde{P}(z)W_1^{-1}(z)(W_2(z))^* + W_2(z)(W_1^{-1}(z))^*(\tilde{P}(z))^* > 0 \quad \forall z \in \partial\mathbf{D}.$$

Multiplying on the left by $(I-P)$ and on the right by $(I-P)^*$ (another congruence transformation) we get

$$P(z)D(z)(P(z))^* + G(z)(P(z))^* - P(z)G(z) - D(z) < 0 \quad \forall z \in \partial\mathbf{D},$$

with, for all $z \in \partial\mathbf{D}$, $D(z) = \mathrm{He}(\ (W_1(z))^{-1}(W_2(z))^*)$, and $G(z) = \mathrm{Sh}(\ (W_1(z))^{-1}(W_2(\mathrm{e}^{\mathrm{j}\theta}))^*)$. It is easy to check that, for all $z \in \partial\mathbf{D}$, $D(z) = (D(z))^*$ and $G(z) = -(G(z))^*$, and condition *(ii)* implies that $D(z) > 0$ for all $z \in \partial\mathbf{D}$. Since the left-hand side and $D(z)$ are both continuous over the compact set $\partial\mathbf{D}$, it follows that, for $\alpha \in (0,1)$ close enough to 1 and for all $z \in \partial\mathbf{D}$,

$$P(z)D(z)(P(z))^* + G(z)(P(z))^* - P(z)G(z) - \alpha D(z) < 0.$$

Since $\alpha \in (0,1)$, this implies that $\|P\|_{\hat{\mu}} < 1$. This completes the proof of the implication (d)⇒(a). The remaining assertion follows directly from the same congruence transformations and the proofs of the implications (b)⇒(c) and (c)⇒(d). The proof of Theorem 3.2 is complete.

Acknowledgment

The authors wish to thank Jie Chen for helpful discussions, as well as Associate Editor Anders Rantzer and the anonymous referees for their many valuable suggestions.

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