ENEE729p. Home assignment 1, due in class on October 2, 2019.

Instructors: Alexander Barg & Ohad Elishco

Each problem is 10 points. Only 3 problems (to be determined) will be graded for correctness. To earn full credit, submit all solutions. If not all submitted, the credit will be reduced proportionally.

Problem 1 (a) Prove the following extension of Chebyshev’s inequality: let \( \psi \) be a positive-valued function on \( \mathbb{R} \) that is nondecreasing on \( \mathbb{R}_+ \); suppose that \( X \) is an RV such that \( E\psi(X) < +\infty \), then for any \( a \geq 0 \) s.t. \( \psi(a) > 0 \)
\[
P(X > a) \leq \frac{E(\psi(X))}{\psi(a)}
\]
(once you have the proof, note the special case \( \psi(x) = x^2 \), which is the standard Chebyshev inequality).

(b) Let \( X \) be an RV with zero mean and variance \( \sigma^2 \). Show that \[
P(\max|X| < a) \geq \min\left(\frac{\sigma^2}{a^2}, \frac{2\sigma^2}{a^2 + \sigma^2}\right).
\]
(Hint: Use Chebyshev’s inequality in the form shown above in part (a); choose suitable functions \( \psi \).)

(c) Suppose that \( X \) is an RV such that \( E(X^2) < \infty \) and \( EX > t \geq 0 \). Prove that
\[
P(X > t) \geq \left(\frac{EX - t}{E(X^2)}\right)^2.
\]

Problem 2. (All the parts of this question refer to random Erdős-Rényi graphs \( G = G_{n,p}; 0 \leq p \leq 1 \).) We say that an event \( P \) is monotone increasing if \( P(G_{n,p} \in \mathcal{P}) \geq P(G_{n,p'} \in \mathcal{P}) \) whenever \( p > p' \).

(a) Let \( u, v \in V(G) \) be two vertices. Prove that the event \( \{u \leftrightarrow v\} \) (there is a path between \( u \) and \( v \)) is monotone increasing.

(b) Prove formally that the event \( A \) is increasing if and only if its complement \( A^c \) is decreasing.

(c) Prove that if \( A, B \) are increasing events, then also \( A \cap B \) is increasing.

(d) Give an example of an event which is neither increasing nor decreasing.

Problem 3. A permutation \( \pi \) is a one-to one function from \( \{1, 2, \ldots, n\} \) to \( \{1, 2, \ldots, n\} \). A fixed point of \( \pi \) is an \( i \in \{1, 2, \ldots, n\} \) such that \( \pi(i) = i \). Suppose that \( \pi \) is chosen uniformly at random from the set of all \( n! \) permutations and let \( X \) be the random number of fixed points.

(a) Find \( E(X) \) and \( \text{Var}(X) \)

(b) Prove that \( P(X \geq 2) \leq \frac{1}{2} \).

Problem 4. Let \( \kappa(G) \) be the clique number of \( G = G_{n,p} \) (the size of the largest complete subgraph in \( G_{n,p} \)). Show that \( n^{-2/3} \) is a threshold for the event \( \{\kappa(G) \geq 4\} \).

Problem 5. (Coupling) Let \( P_1, P_2 \) be probability distributions on some space \((\Omega, \mathcal{F})\).

- A distribution \( P \) on \((\Omega \times \Omega, \mathcal{F} \times \mathcal{F})\) is called a coupling of \( P_1, P_2 \) if its marginals are \( P_1 \) and \( P_2 \).
- Let \( X_1 \sim P_1 \) and \( X_2 \sim P_2 \) be two RVs. We say that the random variable \( Y = (X_1, X_2) \) is a coupling of \( X_1 \) and \( X_2 \) if the distribution of \( Y \) is a coupling of the distributions \( P_1 \) and \( P_2 \).
- Define the total variation distance to be
\[
\|P_1 - P_2\|_{\text{TV}} = \sup_{A \in \mathcal{F}} |P_1(A) - P_2(A)|.
\]
1. Let $P_1$ be a uniform distribution on the two-element set $\{0, 1\}$ and let $P_2$ be a uniform distribution on $\{0, 1, 2\}$.
   (a) Compute the total variation distance between $P_1$ and $P_2$.
   (b) Find two different couplings of $P_1$ and $P_2$.
   (c) Let $X_1 \sim P_1$ and $X_2 \sim P_2$. Find a coupling such that $P(X_2 \geq X_1) = 1$.

2. Let $X_1 \sim P_1$ and $X_2 \sim P_2$ where $P_1$ and $P_2$ are discrete distributions on some probability space $(\Omega, \mathcal{F})$ (not necessarily as in Part 1. of this problem above). Let $P$ be a coupling of $P_1$ and $P_2$; consider an RV $Y = (\hat{X}_1, \hat{X}_2)$ with distribution $P$. Show that
   \[ \|P_1 - P_2\|_{TV} \leq 2P(\hat{X}_1 \neq \hat{X}_2). \]

**Problem 6.** Let $G_{n,m,p}$ be a random intersection graph. The goal of this problem is to show that for $p = o\left(\frac{1}{\sqrt{nm}}\right)$ the edges of $G$ are almost independent, i.e., for every edge $(v_i, v_j)$ there is a common element $l$ in their subsets $S_i, S_j$ such that $l$ does not appear in each of the subsets $S_k, k \neq i, j$.

Let $G_{n,m,p}$ be a intersection graph such that $(v_i, v_j)$ is an edge if and only if the subsets $S_i, S_j$ have a common element $l$ that does not appear in $S_k$ for all $k \in \{1, 2, \ldots, n\}\{i, j\}$. Let $P$ be the distribution of $G_{n,m,p}$ and let $Q$ be the distribution of $G_{n,m,p}$. Use coupling and inequality (1) to show that $\|P - Q\|_{TV} = o(1)$.

**Problem 7.** (Conditional expectation) (Reading: [B17], Sec.2.3)

Remember: CONDITIONAL EXPECTATION IS A RANDOM VARIABLE (RV).

**Part 1.** (Read [AS16], Sec.7.3 “Chromatic number”)

The chromatic number of the graph $\chi(G)$ is the minimum number of colors that can be assigned to the vertices so that two ends of every edge are colored differently (no two adjacent vertices share the same color). For instance, $\chi(K_n) = n$ where $K_n$ is the complete graph on $n$ vertices.

Consider the random graph $G_{n,p} = G_{3,1/4}$ and let $\chi$ be its chromatic number (which is an RV). Imagine the triangle and refer to its edges as bottom (b), left (l), and right (r).

Find $X_i, i = 0, 1, 2, 3$ where

$X_0$ is the expected chromatic number when all the edges are still random events;

$X_1$ is the expected chromatic number when it is known whether (b) is present, while (l) and (r) are still random,

$X_2$ is the expected chromatic number when it is known whether (b) and (l) are present, while (r) is still random, and

$X_3$ is the expected chromatic number when we have information about the presence of all the edges (b),(l),(r).

**Part 2.** Let $Y_1, Y_2$ be two discrete i.i.d. RVs with finite expectation. Find $E(Y_1 | Y_1 + Y_2)$.

**Problem 8.** Consider the graph $G(V, E)$, where $V = \mathbb{Z}$ and $(i, j) \in E$ if and only if $|i - j| = 1$. Show that the percolation threshold for it is $p_c(\mathbb{Z}) = 1$. 