ENEE729p. Final exam

Your paper is due in AVW 2361 on Dec.16, 4:00pm (please slide under the door if locked)

All problems will be graded, 10 points each.

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Problem 1. Let $\mathbb{Z}^2$ be the infinite integer lattice with edges drawn between neighboring sites.

To every edge e assign a weight U_e sampled independently from the uniform distribution Unif $(0,1)$. For ordinary percolation, we call an edge open if its weight $U_e \le p$, for some fixed $p \in (0,1)$.

Now we run a different percolation process on $\mathbb{Z}^2$. In the beginning all the edges are closed.

In Step 1, start with the vertex $(0,0)$ and choose the edge incident to it that has the smallest weight U_e among the four edges incident to $(0,0)$ (ties are broken arbitrarily). Declare the chosen edge open. In Step 2, look at the edges incident to the (two) vertices that are connected to the open edge; choose the edge e with the smallest weight among the closed edges incident to these vertices, and declare it open.

In Step n, choose the edge e with the smallest U_e among the closed edges that are incident to all the vertices connected to at least one open edge, and declare it open. Let $U^{(n)}$ be the weight of the edge opened in the nth step.

(a) Prove that $\limsup_{n\to\infty} U^{(n)} \overset{a.s.}{=} p_c$, where p_c is the percolation threshold for $\mathbb{Z}^2$. (Hint: for $p > p_c$ the usual percolation will produce an infinite cluster with prob. 1. What happens once the process defined above reaches this cluster? Give a formal argument.)

(b) Does this statement hold for $\mathbb{Z}^d, d \ge 3$, namely $\limsup_{n\to\infty} U^{(n)} \overset{a.s.}{=} p_c(\mathbb{Z}^d)$?

Problem 2. Consider a standard GW branching process $(X_n)_{n\ge0}$ with $X_0 = 1$ and $X_{n+1} = \sum_{k=1}^{X_n} Z_{n+1}^{(k)}$, where $(Z_n^{(k)}, n \ge 1, k \ge 1)$ is a collection of iid RVs with finite expectation μ and variance σ^2, taking values in $\mathbb{N}_0$.

(a) Prove that $M_n := X_n/\mu^n$ forms a martingale with respect to the natural filtration $(\mathcal{F}_n)_n$ defined by $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$.

(b) Show that $E(X_{n+1}^2|\mathcal{F}_n) = \mu^2 X_n^2 + \sigma^2 X_n$.

(c) Show that M is bounded in L^2 (i.e., $\sup_{n\ge1} EM_n^2 < \infty$) if and only if $\mu > 1$.

(d) Show that for $\mu > 1$, Var$(M_\infty) = \sigma^2/(\mu(\mu-1))$.

Problem 3. Consider a lazy random walk on the cycle C_n (the transition kernel is $\frac{I_n+P}{2}$, where P is the transition matrix of the simple random walk on C_n). We have proved in class that $t_{\text{mix}} = O(n^2)$ by coupling. Here we approach this question using Wasserstein distance.

(a) Let ρ be the graph distance between the vertices of C_n. Show that for most pairs x, y

$$W_\rho(P(x,\cdot), P(y,\cdot)) = \rho(x,y).$$

Also show that for the same pairs and W_ρ-optimal coupling,

$$E[(\rho(X_1,Y_1) - \rho(x,y))^2 | X_0 = x, Y_0 = y] = 0.$$

(b) Now draw C_n on the unit circle in $\mathbb{R}^2$ and let ρ be the Euclidean distance. Show that

$$W_\rho(P(x,\cdot), P(y,\cdot)) \le e^{-\alpha}\rho(x,y)$$

with $\alpha = c/n^2$.

Problem 4. Consider the following Markov chain on $\mathcal{X} = \{0,1\}^n$. Given a vertex $x = (x_1, \ldots, x_n) \in \mathcal{X}$, choose a uniform random $i \in \{1, 2, \ldots, n\}$. The transition from x occurs as follows: we flip x_i (and move to that vertex) if $x_{i+1} = 1$, otherwise x is unchanged (assume that if $i = n$, then x_n is always flipped, i.e., that x_{n+1} is frozen to 1).

(a) Show that the arising Markov chain is irreducible, reversible, and aperiodic. Find the stationary distribution.

(b) Show that the bottleneck ratio satisfies $\Phi^* \leq \frac{1}{n}$.

(c) Show that $t_{\text{mix}} \geq cn^2$ (Hint: Consider the position of the rightmost 1).

Problem 5. Let X be Poisson(λ). Show that for any $t > \lambda$

$$P(X \geq t) \leq e^{-\lambda} \left(\frac{e\lambda}{t} \right)^t.$$

(this is Exercise 2.3.3 in [V18] which also gives a hint).