Problem 1. Let $\mathbb{Z}^2$ be the infinite integer lattice with edges drawn between neighboring sites.

To every edge $e$ assign a weight $U_e$ sampled independently from the uniform distribution $\text{Unif}(0, 1)$. For ordinary percolation, we call an edge open if its weight $U_e \leq p$, for some fixed $p \in (0, 1)$.

Now we run a different percolation process on $\mathbb{Z}^2$. In the beginning all the edges are closed.

In Step 1, start with the vertex $(0,0)$ and choose the edge incident to it that has the smallest weight $U_e$ among the four edges incident to $(0,0)$ (ties are broken arbitrarily). Declare the chosen edge open. In Step 2, look at the edges incident to the (two) vertices that are connected to the open edge; choose the edge among the four edges incident to $(0,0)$, namely $e$, with the smallest weight among the closed edges incident to these vertices, and declare it open.

In Step $n$, choose the edge $e$ with the smallest $U_e$ among the closed edges that are incident to all the vertices connected to at least one open edge, and declare it open. Let $U^{(n)}$ be the weight of the edge opened in the $n$th step.

(a) Prove that $\limsup_{n \to \infty} U^{(n)} \equiv p_c$, where $p_c$ is the percolation threshold for $\mathbb{Z}^2$. (Hint: for $p > p_c$ the usual percolation will produce an infinite cluster with prob. 1. What happens once the process defined above reaches this cluster? Give a formal argument.)

(b) Does this statement hold for $\mathbb{Z}^d, d \geq 3$, namely $\limsup_{n \to \infty} U^{(n)} \equiv p_c(\mathbb{Z}^d)$?

Problem 2. Consider a standard GW branching process $(X_n)_{n \geq 0}$ with $X_0 = 1$ and $X_{n+1} = \sum_{k=1}^{X_n} Z_{n+1}^{(k)}$, where $(Z_n^{(k)}, n \geq 1, k \geq 1)$ is a collection of iid RVs with finite expectation $\mu$ and variance $\sigma^2$, taking values in $\mathbb{N}_0$.

(a) Prove that $M_n := X_n/\mu^n$ forms a martingale with respect to the natural filtration $(\mathcal{F}_n)_n$ defined by $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$.

(b) Show that $E(X_{n+1}^2 | \mathcal{F}_n) = \mu^2 X_n^2 + \sigma^2 X_n$.

(c) Show that $M$ is bounded in $L^2$ (i.e., $\sup_{n \geq 1} EM_n^2 < \infty$) if and only if $\mu > 1$.

(d) Show that for $\mu > 1$, $\text{Var}(M_\infty) = \sigma^2/((\mu-1))$.

Problem 3. Consider a lazy random walk on the cycle $C_n$ (the transition kernel is $\frac{L_0 + P}{2}$, where $P$ is the transition matrix of the simple random walk on $C_n$). We have proved in class that $t_{\text{mix}} = O(n^2)$ by coupling. Here we approach this question using Wasserstein distance.

(a) Let $\rho$ be the graph distance between the vertices of $C_n$. Show that for most pairs $x,y$

$$W_\rho(P(x,\cdot), P(y,\cdot)) = \rho(x,y).$$

Also show that for the same pairs and $W_\rho$-optimal coupling,

$$E[(\rho(X_1, Y_1) - \rho(x,y))^2 | X_0 = x, Y_0 = y] = 0.$$

(b) Now draw $C_n$ on the unit circle in $\mathbb{R}^2$ and let $\rho$ be the Euclidean distance. Show that

$$W_\rho(P(x,\cdot), P(y,\cdot)) \leq e^{-\alpha} \rho(x,y)$$
Problem 4. Consider the following Markov chain on $X = \{0, 1\}^n$. Given a vertex $x = (x_1, \ldots, x_n) \in X$, choose a uniform random $i \in \{1, 2, \ldots, n\}$. The transition from $x$ occurs as follows: we flip $x_i$ (and move to that vertex) if $x_{i+1} = 1$, otherwise $x$ is unchanged (assume that if $i = n$, then $x_n$ is always flipped, i.e., that $x_{n+1}$ is frozen to 1).

(a) Show that the arising Markov chain is irreducible, reversible, and aperiodic. Find the stationary distribution.

(b) Show that the bottleneck ratio satisfies $\Phi^* \leq \frac{1}{n}$.

(c) Show that $t_{\text{mix}} \geq cn^2$ (Hint: Consider the position of the rightmost 1).

Problem 5. Let $X$ be Poisson($\lambda$). Show that for any $t > \lambda$

$$P(X \geq t) \leq e^{-\lambda \left(\frac{e\lambda}{t}\right)^t}.$$  

(this is Exercise 2.3.3 in [V18] which also gives a hint).