MDP 2023, Final Examination. Date due: May 14, 23:59.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.
- All problems will be graded, max score 105 points.

Problem 1: (15 points)

We are considering binomial intersection graphs $G(n, m, p)$ from Lec. 8.

(a) Show that if $p = \omega(n)/(n\sqrt{m})$, where $\omega(n) \to \infty$, then w.h.p. the graph is nonempty, i.e., $|E| \geq 1$.

(b) Take $p = \left(\frac{2\log n + a(n)}{m}\right)^{1/2}$. What is the condition for $a(n)$ that would imply that w.h.p. the graph is *not* complete? Please be precise in your answer.

Problem 2: (20 points)

A random walk $(X_t)_{t\geq 0}$ on a graph $G = (V, E)$ moves from a vertex to one of its neighbors, chosen uniformly. Starting from $X_0 = u$, let $T_u = \min(t : \forall v \in V, X_s = v$ for some $s \leq t)$ be the first time all $v \in V$ have been visited. Denote $\tau(G) := \max_{u\in V} \mathsf{E}T_u$.

(a) Suppose that G is a simple cycle on n vertices, $V = \{0, 1, \ldots, n-1\}$, and $i \sim j$ iff $|i - j| = 1 \pmod n$. Compute $\tau(G)$.

(b) Let G be a line graph on n vertices, i.e., $V = \{1, 2, \ldots, n\}$, and $i \sim j$ iff $|i - j| = 1$. Compute $\tau(G)$.

Problem 3: (20 points)

Let $t_{\text{mix}}^{(\infty)}$ be the mixing time in the ℓ_∞ norm, defined in [LPW17], Sec.4.7. Consider a lazy random walk on the complete graph on n vertices.

(a) Show that for this random walk, $t_{\text{mix}}^{(\infty)} = \Theta(\log n)$.

(b) Show that for the same random walk, the separation distance $s(2) \leq 1/4$ (The separation distance is defined in Lec.19).

Problem 4: Expander mixing lemma (25 points)

Let $f : E \to \mathbb{R}$ be a function on the state space E of a reversible Markov chain (i.e., an $|E|$-dimensional real vector). Denote by P the transition kernel of the chain. Define a scalar product $\langle f, g\rangle_\pi := \sum_{x\in E} f(x)g(x)\pi(x)$, where π is the stationary distribution, so $\mathsf{E}f = \langle f, 1\rangle_\pi$. Note that

$$\mathrm{Var}(f) = \langle f, f\rangle_\pi - \langle f, 1\rangle_\pi^2.$$

(a) Let $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_{|E|}$ be the eigenvalues of P, and let $\gamma = 1 - \max\{|\lambda_2|, |\lambda_{|E|}|\} = 1 - \max\{|\lambda| : \lambda \neq \lambda_1\}$. Show that for any f

$$\mathrm{Var}(P^t f) \leq (1-\gamma)^{2t}\mathrm{Var}(f).$$

(b) Show that the covariance

$$\mathrm{Cov}(P^t f, g) \leq (1-\gamma)^t \sqrt{\mathrm{Var}(f)\mathrm{Var}(g)}.$$

(c) Let $G(V, E)$ be a d-regular graph with $|V| = n$ and let β be the second largest eigenvalue of the adjacency matrix of G. For a pair of (not necessarily disjoint) subsets $S, T \subset V$ let

$$\mathcal{E}(S,T) = \{(x,y) \in S \times T : x \text{ and } y \text{ are adjacent in } G\}$$

be the edge set of the subgraph induced by $S \cup T$. Prove that

$$\left| |\mathcal{E}(S,T)| - \frac{d|S||T|}{n} \right| \leq \beta\sqrt{|S||T|}. \tag{1}$$

Hint: Take $f = \mathbb{1}_S, g = \mathbb{1}_T$ in (b).

Notes: (1) There are several different proofs of inequality (1). You are required to follow the steps outlined above. Submitting a different proof will result in no credit.

Problem 5: (25 points)

In Lecture 26 we defined the degree-k part of a function $f : \{-1,1\}^n \to \mathbb{R}$ as $f^{=k} = \sum_{|S|=k} \widehat{f}(S)\chi_S$ and the k-th Fourier weight of f as $W^k(f) = \sum_{|S|=k} \widehat{f}(S)^2$. Let $D_i f$ be the ith derivative operator.

(a) Show that $\langle f^{=k}, f^{=l}\rangle = W^k(f)\mathbb{1}_{\{k=l\}}$, where $\langle f, g\rangle := 2^{-n}\sum_{x\in\{-1,1\}^n} f(x)g(x)$.

In parts (b)-(e) the function f is assumed Boolean.

(b) Show that $W^1(f) = 1$ implies that $f = \pm\chi_i$ for some $i \in [n]$.

(c) Prove that if $W^0(f) + W^1(f) = 1$, then f depends on at most one coordinate x_i of its argument.

(d) Suppose that $\mathsf{E}[f] = 0$ and let $I_i(f)$ denote the influence of the ith coordinate (Lec.25). Prove that

$$\max_{1\leq i\leq n} I_i(f) \geq \frac{1}{n}$$

(use the Poincaré inequality, which states that $\mathrm{Var}(f) \leq I(f)$).

(e) For a given $i \in [n]$ show that $|\widehat{f}(i)| \leq I_i(f)$ with equality if and only if the following condition holds: The sign of $(D_i f)(x)$ does not depend on x, i.e., either $(D_i f)(x) \geq 0$ for all x or $(D_i f)(x) \leq 0$ for all x.