Main result about the weight distributions

Theorem 7.1:(MacWilliams) $A^{\perp}(x,y)=2^{-k} A(x+y,x-y)$

So $A(x,y)=2^{-n+k} A^{\perp}(x+y,x-y)$

Example: compute the weight enumerator of $\mathscr{H}_3$ from the w.e. of $\mathscr{S}_{3:}$

$A^{\perp}(x+y,x-y)=(x+y)^7+7(x+y)^3(x-y)^4=8x^7+56\ x^4y^3+56x^3y^4+8y^7$

$=2^{-7+4}\ A(x,y)$

Let $f(x_1, x_2, \ldots, x_n)$ be a function

E.g., $f(x_1, x_2, x_3) = x_1 + x_2x_3$; $f(011) = 1$

Let $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i y_i$ be the dot product

Definition 7.1: The *Fourier (Hadamard) transform* of f

$$\widehat{f}(\mathbf{u}) = \sum_{\mathbf{v} \in \mathbb{F}_2^n} (-1)^{(\mathbf{u},\mathbf{v})} f(\mathbf{v})$$

Lemma 7.2: Let $C[n, k]$ be a linear code. Then

$$\sum_{\mathbf{u} \in C^{\perp}} f(\mathbf{u}) = \frac{1}{2^k} \sum_{\mathbf{u} \in C} \widehat{f}(\mathbf{u})$$

Proof:

$$\begin{aligned}
\sum_{\mathbf{u} \in C} \widehat{f}(u) &= \sum_{\mathbf{u} \in C} \sum_{\mathbf{v} \in \mathbb{F}_2^n} (-1)^{(\mathbf{u},\mathbf{v})} f(\mathbf{v}) = \sum_{\mathbf{v} \in \mathbb{F}_2^n} f(\mathbf{v}) \sum_{\mathbf{u} \in C} (-1)^{(\mathbf{u},\mathbf{v})} \\
&= \sum_{\mathbf{v} \in C^{\perp}} f(\mathbf{v}) \sum_{\mathbf{u} \in C} (-1)^{(\mathbf{u},\mathbf{v})} + \sum_{\mathbf{v} \notin C^{\perp}} f(\mathbf{v}) \sum_{\mathbf{u} \in C} (-1)^{(\mathbf{u},\mathbf{v})} \\
&= |C| \sum_{\mathbf{v} \in C^{\perp}} f(\mathbf{v})
\end{aligned}$$

Proof [of the MacWilliams theorem]: take in the lemma $f(\mathbf{u}) = x^{n-\mathrm{wt}(\mathbf{u})} y^{\mathrm{wt}(\mathbf{u})}$
Let $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n)$

$$\hat{f}(\mathbf{u}) = \sum_{\mathbf{v}\in F} (-1)^{u_1 v_1 + \cdots + u_n v_n} \prod_{i=1}^{n} x^{1-v_i} y^{v_i} = \sum_{v_1=0}^{1} \sum_{v_2=0}^{1} \cdots \sum_{v_n=0}^{1} \prod_{i=1}^{n} (-1)^{u_i v_i} x^{1-v_i} y^{v_i}$$

$$= \prod_{i=1}^{n} \sum_{z=0}^{1} (-1)^{u_i z} x^{1-z} y^{z} = \prod_{i=1}^{n} (x + (-1)^{u_i} y) = (x+y)^{n-\mathrm{wt}(\mathbf{u})} (x-y)^{\mathrm{wt}(\mathbf{u})}$$

Then

$$\sum_{\mathbf{x}\in C^{\perp}} f(\mathbf{x}) = \frac{1}{2^k} \sum_{\mathbf{y}\in C} \hat{f}(\mathbf{y})$$

$$\sum_{\mathbf{x}\in C^{\perp}} x^{n-\mathrm{wt}(\mathbf{x})} y^{\mathrm{wt}(\mathbf{x})} = \frac{1}{2^k} \sum_{\mathbf{y}\in C} (x+y)^{n-\mathrm{wt}(\mathbf{y})} (x-y)^{\mathrm{wt}(\mathbf{y})}$$

$$\sum_{w=0}^{n} A_w^{\perp} x^{n-w} y^{w} = \frac{1}{2^k} \sum_{w=0}^{n} A_w (x+y)^{n-w} (x-y)^{w}$$

$$2^k A^{\perp}(x,y) = A(x+y, x-y)$$

∎

Nonbinary codes

Let C be a linear code of length n over $\mathbb{F}_q$

(means that $\mathbf{x},\mathbf{y} \in C \Rightarrow a\mathbf{x}+b\mathbf{y} \in C$)

For instance, $\mathbb{F}_3=\{0,1,2\}$ with operations mod 3

Definition 7.3. Let $\mathbf{x}=(x_1,x_2,...,x_n)$ be a vector. The Hamming weight $wt(\mathbf{x})=|\{i: x_i \neq 0\}|$. The Hamming distance

$$d(\mathbf{x},\mathbf{y})=wt(\mathbf{x}-\mathbf{y})$$

The weight distribution of the code C

$$(A_0,A_1,....,A_n)$$

The weight enumerator $A(x,y)=\sum_{i=0}^{n} A_i x^{n-i}y^i$

Definition 7.4: The dual code $C^{\perp}=\{\mathbf{y}\in (\mathbb{F}_q)^n : \forall_{x\in C} (\mathbf{x},\mathbf{y})=0\}$

where $(\mathbf{x},\mathbf{y})=\sum_{i=1}^{n} x_i y_i$ (operations in $\mathbb{F}_q$)

Theorem 8.4 (MacWilliams): $A^{\perp}(x,y)= q^{-k} A(x+(q-1)y,x-y)$

Both proofs carry over to the general case