Problem 1.

(a) Let \((A_n)_n\) be an arbitrary sequence of events. Are the following relations correct:
\[
P(\limsup_{n \to \infty} A_n) = \lim_{n \to \infty} P(\bigcup_{i=n}^{\infty} A_i)\\
P(\liminf_{n \to \infty} A_n) = \lim_{n \to \infty} P(\bigcap_{i=n}^{\infty} A_i)\]

(b) Let \((A_n)_n\) and \((B_n)_n\) be sequences of events such that \(P(B_n) \to 1\) as \(n \to \infty\). Show that
\[
\lim_{n \to \infty} P(A_n) = \lim_{n \to \infty} P(A_n B_n)
\]
assuming that at least one of these limits exists.

With the same assumptions, suppose that \(\liminf P(A_n) \geq a > 0\), then show that
\[
\lim_{n \to \infty} P(A_n B_n) = 1.
\]

**Solution:**
(a) Both relations are correct. For instance, \(\bigcup_{i=n}^{\infty} A_i\) is a decreasing (with \(n\)) event, and thus
\[
\lim_{n \to \infty} P(\bigcup_{i=n}^{\infty} A_i) = P(\bigcap_{i=1}^{\infty} A_i) = P(\limsup_{n \to \infty} A_n).
\]

(b) It is clear that \(\lim_{n \to \infty} P(A_n) \geq \lim_{n \to \infty} P(A_n B_n)\). To prove the reverse inequality, write
\[
\lim_{n \to \infty} P(A_n) = \lim_{n \to \infty} P(A_n B_n^c) + P(A_n B_n) = \lim_{n \to \infty} P(A_n B_n^c) + \lim_{n \to \infty} P(A_n B_n) \\
\leq \lim_{n \to \infty} P(B_n^c) + \lim_{n \to \infty} P(A_n B_n) = \lim_{n \to \infty} P(A_n B_n).
\]

For the second statement, write
\[
\frac{P(A_n)}{P(A_n B_n)} = \frac{P(A_n B_n) + P(A_n B_n^c)}{P(A_n B_n)} = 1 + \frac{P(A_n B_n^c)}{P(A_n B_n)}
\]
Taking the limits and relying on the assumption about \(A_n\), we obtain the claim (without this assumption, it is not clear that the denominator does not tend to zero).

Problem 2.

Let \((X_n)_n\) be a sequence of independent RVs with
\[
P(X_n = 1) = P(X_n = -1) = \frac{1}{2}(1 - 2^{-n}), \quad P(X_n = 2^n) = P(X_n = -2^n) = 2^{-n-1}.
\]

Show that \(\sum_{n \geq 1} \frac{\text{Var}(X_n)}{n^2} = \infty\) but nevertheless the sequence \((X_n)_n\) satisfies the SLLN, i.e.,
\[
\frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s.} 0
\]
(for the second claim you may attempt to show that \(\sum_{n \geq 1} P(|X_1 + \cdots + X_n| > n\epsilon) < \infty\)).

**Solution:** \(\text{Var}(X) = 1 - 2^{-n} + 2 \cdot 2^{2n} \cdot 2^{-n-1} = 1 - 2^{-n} + 2^n\), so
\[
\sum_{n=1}^{\infty} \frac{\text{Var}(X_n)}{n^2} \geq \sum_{n=1}^{\infty} \frac{2^n}{n^2} > \sum_{n=1}^{\infty} 1 = \infty,
\]
proving the first claim.

Turning to the second claim, notice that $P(|X_n| = 2^n)$ declines rapidly with $n$. Our plan is to show that for the most part, the $X_n$'s will be $\pm 1$, and whatever values $\pm 2^i$ appear, will be concentrated among the first (constant) number of indices in the set $\{1, 2, \ldots, n\}$.

Since $P(|X_n| = 2^n) = 2^{-n}$, we have that $P(|X_n| = 2^n$ i.o.) = 0, so with probability 1 there is a number $N = N(\omega)$ such that the realization $(X_n)_{n \geq 1}$ satisfies $|X_n| = 1$ for all $n > N$. The values $2^i$ that occur before $N$ will not play a role in the sum $\frac{X_1 + \cdots + X_n}{\sqrt{n \log n}}$ because they add to a constant which, divided by $n$, tends to 0 as $n \to \infty$. For $n \geq N$ the sequence $(X_n)_{n \geq 1}$ is a realization of the symmetric random walk, and it is easily seen that
\[
S_{N,n} = \sum_{i=1}^n X_i
\]
has the property that $-\frac{S_{N,n}}{\sqrt{n \log n}} \to 0$ a.s. To show this, notice that if $Y_n$ are independent RVs with $P(Y_n = 1) = P(Y_n = -1) = 1/2$, $EY = 0$, then $\text{Var} \frac{Y_n}{\sqrt{n \log n}} = \frac{1}{n \log n}$, and
\[
\sum_{n=1}^\infty \frac{1}{n \log^2 n} < \infty,
\]
so SLLN applies to the sequence $Y_n$, which shows that from $N$ onward, the sum $|S_{N,n}|$ stays in the neighborhood of $\sqrt{n \log n}$ with probability one, i.e., $\frac{|S_{N,n}|}{n} \to 0$. Thus, overall the sequence $(X_n)_{n \geq 1}$ satisfies the SLLN.

(It is easy to show, after the conclusion about $(Y_n)$, that $\sum_{n \geq 1} P(|X_1 + \cdots + X_n| > \epsilon n) < \infty$ but we can also manage without this claim.)

Among the mistakes made in this solution was an attempt to apply SLLN right away to the sequence $(X_n)_{n \geq 1}$ based on $E X = 0$, missing the fact that the RVs $X_n$ are not identically distributed.

**Problem 3.**

(a) Let $X_n, n \geq 1$ be i.i.d. Bernoulli RVs with $P(X = 0) = P(X = 1) = 1/2$. What is the distribution of the RV $Y = \sum_{n=1}^\infty X_n/2^n$?

(b) Let $X$ and $Y$ be independent RVs on $(\Omega, F, P)$ with CDFs $F_X(x)$ and $F_Y(x)$. What is the CDF of the RV $Z = XY$? (recall that for an RV $U$ we have $P(a \leq U \leq b) = \int_a^b dF_U(x)$).

**SOLUTION:** (a) The following argument is somewhat informal: Let $Y_n = \sum_{k=1}^n X^k/2^k$, then straightforward induction shows that $Y_n$ takes the values $i/2^n, i = 0, 1, \ldots, 2^n-1$ with probability $2^{-n}$ each. Then
\[
P(Y_n \leq m/2^n) = m/2^n.
\]
For a given $n$ and $x \in [0,1]$, $\frac{m}{2^n} \leq x < \frac{m+1}{2^n}$, we find $P(Y_n \leq x) = \frac{m}{2^n}$, where $m = 0, 1, \ldots, 2^n-1$. As $n$ increases, we approximate the value of $x$ more and more closely, finding in the limit that $P(Y \leq x) = x$, i.e., $Y \sim \text{Unif}[0, 1]$.

(These arguments suffice for full credit).

The next part is for your information only. To construct a formal argument, recall our discussion in Lectures 1 and 2 of the course. Our sample space $\Omega$ is the set of all infinite sequences of coin tosses, endowed with a probability measure, which is constructed as follows. Suppose that the first $n$ tosses yield $x_1, x_2, \ldots, x_n$, and consider the events $E_n(x_1, \ldots, x_n)$ formed of all $\omega$'s that start with $x_1, \ldots, x_n$ (such events are called cylinder sets). The collection of all cylinder sets $E_n$ forms a $\sigma$-algebra $A_n$, and moreover $A_1 \subset A_2 \subset \ldots$. The set $F = \cup_{n \geq 1} A_n$ is the $\sigma$-algebra of events in our experiment. By Kolmogorov's existence theorem there is a unique measure $\mu$ on $(\Omega, F)$ that is consistent with the distributions on $A_n$ for all finite values of $n \in \mathbb{N}$. Now let us define a map $\phi : \Omega \to [0,1]$ given by $f(\omega) = \sum_{k=1}^\infty X_k(\omega)/2^k$, and note that $f$ is a measurable function. Now (with some more work), the coin tossing measure $\mu$ assigns the probability $y - x$ to any pair of numbers $0 \leq x \leq y \leq 1$, and this measure extends uniquely to the Borel $\sigma$-algebra on $[0,1]$ by Carathéodory's theorem, giving the Lebesgue measure on $[0,1]$. Thus, the RV $Y$ is uniform on $[0,1]$.

Some students wrote that the distribution we obtain is a Cantor distribution. The (standard) Cantor distribution is obtained if instead of the above function $f$ we take $g(\omega) = \sum_{k \geq 1} 2X_k(\omega)/3^k$, and the CDF of this distribution is
constant, for instance, on the interval \((1/3, 2/3)\). The distribution discussed above is not constant, so this answer is incorrect.

(b) We have \(XY \leq z\), translating into \(Y \leq z/X\) if \(X > 0\) and \(Y \geq z/X\) if \(X < 0\), and so

\[
P(Z \leq z) = P(XY \leq z) = P(Y \leq z/X) = \int_{-\infty}^{0} (1 - F_Y(z/x))dF_X(x) + \int_{0}^{\infty} F_Y(z/x)dF_X(x).
\]

**Problem 4.**

(a) For i.i.d. RVs \(X_1, \ldots, X_n\) with CDF \(F\) find the CDF of the RV \(Y_n := \max(X_1, \ldots, X_n)\).

For the remaining parts of the problem suppose that \(X_i, i = 1, \ldots, n\) take values on \(\mathbb{N}_0\) and

\[P(X = k) = 1/2^{k+1}, k \geq 0.\]

(b) Show that for some constant \(c > 1\), the probability \(P(Y_n \geq c \log_2 n) \to 0\), where \(Y_n\) is defined in Part (a).

(c) Therefore prove that for some positive constants \(A < B\) (which may depend on \(n\)), the expectation of \(Y_n\) satisfies

\[A \log_2 n \leq EY_n \leq B \log_2 n.\]

(d) Show that the sequence \(Y_n/n\) converges to 0 in probability (please do not rely on the result of part (e)).

(e) Show that the sequence \(Y_n/n\) converges to 0 a.s.

**Solution:**

(a) If \(\max_i X_i \leq x\), then every RV \(X_i \leq x\), so

\[F_{Y_n}(x) = P(Y_n \leq x) = \prod_{i=1}^{n} P(X_i \leq x) = F_X(x)^n.\]

(b) For a positive integer \(m\)

\[F_X(m) = \sum_{k=0}^{m} \frac{1}{2^{k+1}} = \frac{2^m - 1}{2^{m+1}},\]

and thus

\[F_{Y_n}(x) = \left(1 - \frac{1}{2^{m+1}}\right)^n, \text{ where } m = \lfloor x \rfloor.\]

Then, taking \(k = \lfloor c \log_2 n \rfloor\)

\[P(Y_n \leq k) = F_{Y_n}(k)^n = \left(1 - \frac{1}{2^{k+1}}\right)^n = \left(1 - \frac{1}{n^{c'}}\right)^n \to 1\]

since \(c' > 1\), proving the claim.

(c) Below we take \(k = \lfloor c \log_2 n \rfloor\). We have

\[EY_n = E[Y_n \cap \{Y_n \leq k - 1\}] + E[Y_n \cap \{Y_n \geq k\}]\]

The first term is at most \(k \leq c \log_2 n\), which gives us the needed lower bound. To derive an upper bound, we estimate the second term:

\[E[Y_n \cap \{Y_n \geq k\}] = \sum_{i=k}^{\infty} iP(Y_n = i) = \sum_{i=0}^{\infty} (k+i)P(Y_i \geq i + k) = \sum_{i=0}^{\infty} (k+i)(1 - (1 - 2^{-(i+k+1)})^n).\]

Since \((1 - 2^{-m})^n \geq 1 - n2^{-m}\), we continue

\[E[Y_n \cap \{Y_n \geq k\}] \leq n \sum_{i=0}^{\infty} (k+i)2^{-k-i-1} = n(k+2)2^{-k-1} \leq 2n(c \log_2 n)n^{-c} \leq B' \log_2 n,\]

where \(B' = 2c\). Taking this together with the first term, we obtain the upper bound.

**Remark:** The problem statement mentioned that \(A\) and \(B\) could depend on \(n\). This is incorrect (meaningless), and should not have been included. Because of this mistake, you will get full credit for Part (c) if your solution relies on constants that depend on \(n\) if the logic is correct otherwise.
(d) Likewise, the second moment $EY^2$ is bounded as $\text{Var}(Y_n) \leq EY^2_n \leq C \log^2 n$, and then by Chebyshev’s inequality
\[
P(Y_n > ne) \leq \frac{C \log^2 n}{n^2 \epsilon^2} \to 0
\]
proving convergence in probability.

(e) Further, from the last displayed equation, $\sum_n P(Y_n > ne)$ is finite, and thus $P(Y_n > \epsilon \text{ i.o.}) = 0$, proving a.s. convergence.

**Problem 5.**

An urn contains $N$ Red balls. The following step is repeated indefinitely: Take a random ball out of the urn, and replace it with a ball that is Blue with probability $p$ and Red with probability $1 - p$ independently of the color of the removed ball. Let $Y_n$, $n \geq 1$ be the number of Red balls in the urn after $n$ such steps.

For all values of $p \in [0, 1]$ please give answers to the following two questions.

(a) Find $p_{ij} = P(Y_{n+1} = j|Y_n = i)$ for all $i, j = 0, 1, \ldots, N$. Does the sequence $(Y_n)_n$ form a Markov chain?

(b) Find $\lim_{n \to \infty} P(Y_n = k)$ for all $k = 0, 1, \ldots, N$ (rather than computing, guess and verify the answer. The pivot is provided by the fact that the process “forgets” its starting state).

**Solution:** (a) The sequence forms a Markov chain for all $p$ as will be shown by computing the transition probabilities.

If $p = 0$ then $Y_n = N$ with probability 1 for all $n$, and $p_{ij} = 1$ if $i = j = N$. The limiting distribution is
\[
\pi_k = 1_{\{k=N\}}, k = 0, 1, \ldots, N.
\]

If $p = 1$, then the transitions are $p_{N,N-1} = 1, p_{i,i-1} = \frac{i}{N}, 1 \leq i \leq N$, and $p_{00} = 1; p_{ij} = 0$ for all other $i, j$. Thus the states $1, \ldots, N$ are transient, while the state 0 is recurrent (and absorbing). Therefore the limiting distribution is
\[
\pi_k = 1_{\{k=0\}}, k = 0, 1, \ldots, N.
\]

For $0 < p < 1$ the transition probabilities are $p_{ij} = 0$ if $|i - j| \geq 2$, and
\[
p_{k,k+1} = \frac{N-k}{N} (1-p), 0 \leq k \leq N-1
\]
\[
p_{k,k} = \frac{k}{N} (1-p) + \frac{N-k}{N} p, 0 \leq k \leq N
\]
\[
p_{k,k-1} = \frac{k}{N} p, 1 \leq k \leq N
\]
and thus the process forms a homogeneous Markov chain (a random walk with reflecting screens).

For part (b) guess that the pmf $\pi_k = \binom{N}{k} p^{N-k}(1-p)^k, k = 0, \ldots, N$ is stationary. The guess is justified by the fact in the limit the count of balls is independent of the starting state, and is governed by independent choice with probability $P(\text{Red}) = 1 - p$. After that check the equations $\pi P = \pi$. Indeed, we have
\[
\pi_k (1-p_{kk}) = \pi_{k-1} p_{k-1,k} + \pi_{k+1} p_{k+1,k}, 1 \leq k \leq N-1
\]
\[
\pi_0 (1-p) = \pi_1 \frac{p}{N}, \pi_N p = \pi_{N-1} \frac{1-p}{N}
\]
and these equations are checked directly with the given values of $\pi_k$. From the ergodic (convergence) theorem for irreducible aperiodic Markov chains, this is also the limiting distribution.