ENEE620. Final examination, 12/18/2023. Instructor: A. Barg

- Please submit your work to ELMS Assignments as a single PDF file by Dec.18, **6:00pm** EST.
- Each problem is 10 points. Max score=50 points
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1. Let $U_n, n \ge 1$ be independent uniformly distributed random variables on $[0,1]$. Define a sequence of random variables $(X_n)_{n\ge 0}$ such that $X_0 = 0$, and for $n \ge 1$

$$X_n = \max\{X_{n-1}, (X_{n-1} + U_n)/2\}.$$

(a) Does the limit of the sequence $(X_n)_{n\ge 0}$ exist in probability? If yes, what is the limiting random variable?
(b) Does the almost sure limit of the sequence $(X_n)_{n\ge 0}$ exist?
(c) Does the mean square limit of the sequence $(X_n)_{n\ge 0}$ exist?

Problem 2.

Let $B(t), t \ge 0$ be a standard Brownian motion. Fix a value of $t > 0$. For all $n \ge 1$, partition the segment $[0,t]$ into an increasing number $k(n)$ nonoverlapping intervals $0 = t_0^{(n)} \le t_1^{(n)} \le \cdots \le t_{k(n)-1}^{(n)} \le t_{k(n)}^{(n)} = t$ and suppose that the length of the segments vanishes as n increases, i.e.,

$$\lim_{n\to\infty} \max_{1\le j\le k(n)} (t_j^{(n)} - t_{j-1}^{(n)}) = 0.$$

(a) Show that

$$\lim_{n\to\infty} \sum_{j=1}^{k(n)} (B(t_j^{(n)}) - B(t_{j-1}^{(n)}))^2 = t$$

in the mean-square sense. Hint: Use $t = \sum_{j=1}^{k(n)} (t_j - t_{j-1})$ and independence of increments.

(b) Suppose moreover that

$$\sum_{n=1}^{\infty} \sum_{j=1}^{k(n)} (t_j^{(n)} - t_{j-1}^{(n)})^2 < \infty,$$

then show that the convergence in (a) holds almost surely.

Problem 3.
Consider a branching process $X_n, n \ge 0$ with $X_0 = 1$.

(a) Suppose that the offspring distribution is $P(Z = i) = (1-p)^i p, i \ge 0$, where $0 < p < 1$. Show that the probability of extinction is

$$P_{\text{ex}} = \begin{cases} \frac{p}{1-p} & \text{if } p < \frac{1}{2} \\ 1 & \text{if } p \ge \frac{1}{2}. \end{cases}$$

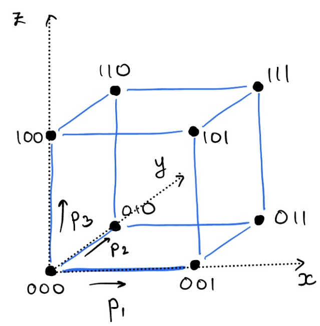
(b) Now suppose that $P(Z = i) = e^{-\lambda} \frac{\lambda^i}{i!}, i \ge 0$. Find the probability of extinction if $(1)\lambda = \ln 2$, $(2)\lambda = 2\ln 2$.

(c) In the setting of part (b), suppose that $\lambda > 1$ and let E be the extinction event. Show that

$$P(E) - P(\tau \le n) \le (\lambda P(E))^n,$$

where τ is the random time to extinction.

Problem 4.
Consider the cube C in $\mathbb{R}^3$ with vertices (a_1, a_2, a_3) where each of the three coordinates is either 0 or 1 (i.e., (000), $(001), \ldots, (111)$). A dicrete-time random walk on the vertices of C starts at (000) and moves along an edge to one of the three neighbors of (000). From any vertex v, inclduing (000), the random walk moves to one of its three neightbors using the following rule: it moves parallel to the x-axis with probability p_1, parallel to the y-axis with probability p_2, and parallel to the z-axis with probability p_3. Assume that $0 < p_i < 1, i = 1,2,3$ and $p_1 + p_2 + p_3 = 1$.



(a) Write out the transition matrix of the arising Markov chain.

(b) Find the limiting distribution of this Markov chain.

(c) Find the expected number of visits to (111) before the first return to (000).

Problem 5.

Let $X_1, \dots, X_n$ be independent Gaussian random variables with $X \sim \mathcal{N}(0,1)$, and let $S_k = X_1 + \cdots + X_k, k \geq 1$, denote their partial sums. Assume that $S_0 = 0$ and let $\Phi(x) = P(X \leq x), x \in \mathbb{R}$. Form a filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_k = \sigma(X_1, \dots, X_k), k \geq 1$. Prove that for every real number a, the sequence $(Y_k, \mathcal{F}_k), k = 0, 1, \dots, n$ defined by

$$Y_k = \Phi\Big(\frac{a - S_k}{\sqrt{n-k}}\Big)$$

forms a (finite) martingale.