ENEE620. Final examination, December 16-17, 2020. Instructor: A. Barg

- Please submit your work as a single PDF file to ELMS/Canvas Assignments by 12/17, 4pm US Eastern time.
- Max score=85 points
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1. (20pt) For $n \in \mathbb{N}$ consider probability mass functions $p_{k,n}, k = 1, \dots, n$ given by

$$p_{k,n}(-\sqrt{n}) = p_{k,n}(\sqrt{n}) = \frac{1}{2n};\ p_{k,n}(0) = 1 - \frac{1}{n}.$$

Let $X_{k,n}, k = 1, \dots, n$ be i.i.d. RVs with pmf $p_{k,n}$ and define

$$S_n = \sum_{k=1}^{n} X_{k,n}, \quad n = 1, 2, \dots$$

(a) For each $n = 1, 2, \dots$ find $EX_{k,n}, \text{Var}(X_{k,n})$ for all $k = 1, \dots, n$.

(b) Does the sequence $\{\frac{S_n}{n}, n = 1, 2, \dots\}$ converge in probability? If yes, what is the limiting RV?

(c) Consider the sequence

$$\Big\{R_n = \frac{S_n}{\sqrt{\text{Var}(S_n)}}, n = 1, 2, \dots\Big\}.$$

Does it converge in distribution? If yes, what is the limiting RV? (One way to proceed is to use characteristic functions; in particular, what is this function for the Poisson distribution?).

Problem 2. (15pt) Let $(N_t)_{t\ge 0}$ be a Poisson process PP(λ). Consider a process

$$X_t = \xi \cdot (-1)^{N(t)}, \quad t \ge 0$$

where ξ is a Bernoulli RV independent of N and such that $P(\xi = 1) = P(\xi = -1) = 1/2$ (for a given t we flip the coin to find ξ and then compute X_t). Show that $EX_t = 0$ and $E[X_s X_t] = e^{-2\lambda|t-s|}$.

Problem 3. (15pt) Let $X_1, X_2, \dots$ be a sequence of i.i.d. nonnegative RVs with a continuous CDF $F(x)$. Let $Z_1 = 1$, $Z_k = \min\{n > Z_{k-1} : X_n \ge \max(X_1, \dots, X_{n-1})\}, k \ge 1$ be the sequences of *records*.

Does the sequence of RVs $(Z_k)_{k\ge 1}$ form a Markov chain? If yes, find the transition probability matrix of this Markov chain.

Problem 4. (20pt) Let X and Y be two Gaussian RVs with $EX = EY = 0, EX^2 = EY^2 = 1$, and $E(XY) = \rho$. Show that

(a) $E[\max(X, Y)] = \sqrt{(1-\rho)/\pi}$

(b) $E(X|Y) = \rho Y$, $\text{Var}(X|Y) = 1 - \rho^2$

(c) $E(X|X+Y = z) = z/2$, $\text{Var}(X|X+Y = z) = (1-\rho)/2$

(d) $E(X+Y|X > 0, Y > 0) = 2\sqrt{2/\pi}$.

Problem 5. (15pt) Let $Z_i, i \ge 1$ be a collection of i.i.d. RVs with $P(Z = -1) = 3/4$ and $P(Z = c) = 1/4$, where c is a real number. Let $X_n = X_0 + Z_1 + Z_2 + \dots + Z_n, n \ge 1$, where $X_0 = 5$.

(a) Find a value of c such that the sequence $(X_n)_n$ forms a martingale.

(b) For the value of c found in Part (a), is there an RV X such that $X_n \overset{\text{a.s.}}{\to} X$? Please justify your answer.

(c) For the value of c found in Part (a), is it true that $P(\cup_{n\ge 1}\{X_n = 0\}) = 1$? Please justify your answer.