READ THIS:
- The exam consists of five problems. Each problem is 10 points. Max score = 50 points.
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- You do not have to copy problem statements into your paper.
- If applicable, clearly identify the answer to the question.

Problem 1. We are given \( n \) independent RVs \( X_1, \ldots, X_n \), each of which is uniformly distributed on the unit interval \([0, 1]\). Define \( Y_n \triangleq \prod_{i=1}^{n} X_i \), \( n = 1, 2, \ldots \).

(a) Find \( EY_n \), \( n = 1, 2, \ldots \).
(b) Find \( E(Y_{n+1}|Y_n) \), \( n = 1, 2, \ldots \).
(c) Find \( E(Y_{n+1}|Y_1, Y_2, \ldots, Y_n) \), \( n = 1, 2, \ldots \).
(d) Does the sequence \( (Y_n, n = 1, 2, \ldots) \) converge in probability; if yes, what is the limit? Does it also converge a.s.?

Solution: (a) \( EY_n = 2^{-n} \).

(b) Since \( Y_{n+1} = Y_n X_{n+1} \), and \( X_{n+1} \) is independent of \( Y_n \), we obtain
\[
E(Y_{n+1}|Y_n) = E(Y_n X_{n+1}|Y_n) = Y_n E(X_{n+1}|Y_n) = Y_n E(X_{n+1}) = \frac{1}{2} Y_n.
\]

(c) \[ E(Y_{n+1}|Y_1, \ldots, Y_n) = E(Y_n X_{n+1}|Y_1, \ldots, Y_n) = Y_n E(X_{n+1}) = \frac{1}{2} Y_n. \]

(d) Let \( \epsilon > 0 \), then
\[
P(Y_n > \epsilon) \leq EY_n/\epsilon = \epsilon^{-1} 2^{-n} \rightarrow 0.
\]
Moreover, also \( \sum_{n=1}^{\infty} P(Y_n > \epsilon) < \infty \), and thus \( P(Y_n > \epsilon \text{ i.o.}) = 0 \), or \( Y_n \xrightarrow{a.s.} 0 \).

Problem 2. (a) We consider a martingale \( Z_n, n = 1, 2, \ldots \) with respect to the natural filtration \( \mathcal{F}_n = \sigma(Z_1, \ldots, Z_n) \).
Show that for \( 1 \leq k < n \)
\[
E(Z_n|Z_1, \ldots, Z_k) = Z_k.
\]
(b) Let \( X_n, n = 1, 2, \ldots \) be a martingale with respect to the natural filtration \( \sigma(X_1, \ldots, X_n) \), and let \( Y_n, n = 1, 2, \ldots \) be another, independent martingale with respect to the natural filtration \( \sigma(Y_1, \ldots, Y_n) \). Show that the sequence \( Z_n = X_n + Y_n \) forms a martingale with respect to \( \sigma(Z_1, \ldots, Z_n), n \geq 1 \) (use the tower property of conditional expectations). If the independence assumption is removed, is this statement still true? Prove or give a counterexample.

Solution: (a)
\[
Z_k = E(Z_{k+1}|Z_k, \ldots, Z_1) = E(E(Z_{k+2}|Z_{k+1}, \ldots, Z_1)|Z_k, \ldots, Z_1) = E(Z_{k+2}|Z_k, \ldots, Z_1),
\]
and generally,
\[
E(Z_n|Z_k, \ldots, Z_1) = E(E(Z_{n+1}|Z_n, \ldots, Z_1)|Z_k, \ldots, Z_1) = E(Z_{n+1}|Z_k, \ldots, Z_1)
= \cdots = E(Z_{k+1}|Z_k, \ldots, Z_1) = Z_k.
\]

(b) Since \( X_n, Y_n \) are integrable by assumption, the RV \( Z_n \) is also integrable. Further,
\[
E(X_n|Z_1, \ldots, Z_{n-1}) = E(E(X_n|Z_1, \ldots, Z_{n-1}, X_1, \ldots, X_{n-1})|Z_1, \ldots, Z_{n-1})
= E(E(X_n|X_1, \ldots, X_{n-1})|Z_1, \ldots, Z_{n-1}) = E(X_{n-1}|Z_1, \ldots, Z_{n-1}),
\]
where on the last line we used independence (given \( X_1^{n-1}, X_n \) is independent of \( Z_1^{n-1} \)). In the same way,
\[
E(Y_n|Z_1, \ldots, Z_{n-1}) = E(Y_{n-1}|Z_1, \ldots, Z_{n-1})
\]
and adding, we obtain
\[
E(X_n + Y_n|Z_1, \ldots, Z_{n-1}) = E(X_{n-1} + Y_{n-1}|Z_1, \ldots, Z_{n-1}) = X_{n-1} + Y_{n-1}
\]
as required.
Thus, $0 = \sum_{i=1}^{N(t)} X_i$.

**Problem 3.** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with common mean 1, and let $N(t), t \geq 0$ be a Poisson process with rate 2 that is independent of the RVs $X_i, i \geq 1$. Define a random process $Y(t), t \geq 0$ by setting $Y(0) = 0$ and for $t > 0$

$$Y(t) = \sum_{i=1}^{N(t)} X_i.$$

(a) Find the mean function $EY(t)$ of the process $Y$.
(b) Show that the process $Y$ has independent increments.

**Solution:** (a) $EY(t) = E(E(Y(t)|N(t))) = E\left(E\left(\sum_{i=1}^{N(t)} X_i | N(t)\right)\right) = E\left(\sum_{i=1}^{N(t)} EX\right) = 2t \cdot 1 = 2t$.

(b) Independence of increments follows from the independence of increments of the Poisson process. Indeed, let us fix the values $0 = t_0 < t_1 < t_2 < \cdots < t_n$, and denote $\Delta_k \triangleq Y(t_k) - Y(t_{k-1}), k = 1, \ldots, n$. Clearly $\Delta_k = \sum_{i=N(t_{k-1})+1}^{N(t_k)} X_i$. Given the increments $N(t_k) - N(t_{k-1})$, the RVs $\Delta_k$ are mutually conditionally independent. Moreover, $\Delta_k$ is conditionally independent of all the other $\Delta_j$’s given the starting count $N(t_k)$ and the value $N(t_k) - N(t_{k-1})$. So, $\Delta_k$ is conditionally independent of all the other $\Delta_j$’s given $N(t_1) - N(0), N(t_2) - N(t_1), \ldots, N(t_k) - N(t_{k-1})$. Since by the properties of the Poisson process, these increments are independent, we conclude that $\Delta_1, \ldots, \Delta_n$ are independent as well.

**Problem 4.** Let $X$ be a Bernoulli RV, $P(X = 1) = 1 - P(X = -1) = 1/2$, and let

$$X_n \triangleq \begin{cases} X & \text{with probability } 1 - \frac{1}{n} \\ e^n & \text{with probability } \frac{1}{n}. \end{cases}$$

Is it true that
(a) $X_n \xrightarrow{P} X$;
(b) $X_n \xrightarrow{d} X$ (to receive credit, justify your answer directly, without appealing to other modes of convergence);
(c) $\lim_{n \to \infty} E[(X_n - X)^2] = 0$ ?

In each case justification is required.

**Solution:** (a) We write

$$P(|X_n - X| > \epsilon) = \left(1 - \frac{1}{n}\right)P(|X_n - X| > \epsilon|X_n = X) + \frac{1}{n}P(|X_n - X| > \epsilon|X_n = e^n)$$

$$= \frac{1}{n}P(|e^n - X| > \epsilon) \leq \frac{1}{n},$$

which shows that $X_n \xrightarrow{P} X$.

(b) part (a) already implies this, yet, we set out to prove this independently. We have

$$F_X(x) = \begin{cases} 0 & x < -1 \\ 1/2 & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \quad \text{and} \quad F_{X_n}(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{2}(1 - \frac{1}{n}) & -1 \leq x < 1 \\ 1 - \frac{1}{n} & 1 \leq x < e^n \\ 1 & x \geq e^n \end{cases}$$

Thus, $F_{X_n}(x) \to F_X(x)$ for all real $x$ at which $F_X$ is continuous.

(c)

$$E(X_n - X) = \left(1 - \frac{1}{n}\right)E((X_n - X)|X_n = X) + \frac{1}{n}E((X_n - X)|X_n = e^n)$$

$$= \frac{1}{n} \left(\frac{(e^n + 1)^2}{2} + \frac{(e^n - 1)^2}{2}\right) \to \infty,$$
so this is false.

Problem 5. Assume that $X$ and $Y$ are jointly Gaussian processes with zero mean, autocorrelation functions $R_X(t) = R_Y(t) = e^{-|t|}$ and cross-correlation function $R_{XY}(t) = \frac{1}{2}e^{-|t-3|}$. Assume further that for any $n \geq 1$, any real $t_1, \ldots, t_n$ and any $s > 0$, the distribution of the vector $(X_{t_1+s}, \ldots, X_{t_n+s}, Y_{t_1+s}, \ldots, Y_{t_n+s})$ does not depend on $s$.
(a) Find the autocorrelation function of the random process $Z(t) = \frac{1}{2}(X(t) + Y(t))$, $t \in \mathbb{R}$.
(b) Is $Z(t)$ a (strict-sense) stationary process?
(c) Find the variance of the RV $X(1) - 3Y(2)$. Then find $P(X(1) < 3Y(2) - 1)$ (express your answer in terms of the standard normal CDF $\Phi$).

Solution: (a)

\[ R_Z(s, t) = E[(1/4)(X(s) + Y(s))(X(t) + Y(t))] = \frac{1}{4}(R_X(s-t) + R_Y(s-t) + R_{XY}(s-t) + R_{YX}(s-t)). \]

Thus, $R_Z(s, t)$ depends only on $s - t$. Using $R_{YX}(y, x) = R_{XY}(x, y)$, we obtain

\[ R_Z(t) = \frac{1}{4}(2e^{-|t|} + \frac{1}{2}e^{-|t-3|} + \frac{1}{2}e^{-|t+3|}). \]

(b) Since $EZ(t) = 0$ and $R_Z(s, t)$ depends only on $s - t$, we conclude that $Z$ is WSS. Further, $Z$ by definition is a Gaussian process, and for such processes, WSS implies stationarity.

(c) We have $P(X(1) < 3Y(2) - 1) = P(X(1) - 3Y(2) < -1)$, and $X(1) - 3Y(2)$ is a zero-mean Gaussian RV with variance

\[ \sigma^2 = \text{Var}(X(1) - 3Y(2)) = R_X(0) - 6R_{XY}(-1) + 9R_Y(0) = 1 + 9 - 3e^{-4}. \]

Then $(1/\sigma)(X(1) - 3Y(2)) \sim \mathcal{N}(0, 1)$, and $P\left(\frac{X(1) - 3Y(2)}{\sigma} < -1/\sigma\right) = \Phi(-\frac{1}{\sigma}).$