ENEE620-23. Home assignment 5. Date due December 5, 11:59pm EDT.

Instructor: A. Barg

Please submit your work as a single PDF file to ELMS (under the "Assignments" tab)
• Papers submitted as multiple pictures of individual pages are difficult for grading and will not be accepted.
• Justification of solutions is required.
• Each problem is worth 10 points.

Problem 1. An $N \times N$ checkers board is covered with checkers pieces of k different colors, meaning that one piece is placed on each square. Now, in the regular checkers game the pieces are of two colors, but in this problem they can have $k \geq 2$ different colors. Let X_0 be the starting coloring (allocation of colors to the pieces). The random process of color evolution proceeds as follows: choose a random piece (with uniform distribution), choose uniformly its neighbor, and replace the chosen piece with a piece with the color of the chosen neighbor.

(a) Fix a color 0 and let Y_n be the proportion of 0-colored pieces at time n. Prove that $(Y_n)_n$ forms a martingale with respect to $(\mathcal{F}_n)_n$, where $\mathcal{F}_n = \sigma(X_0, X_1, \dots X_n)$.

(b) Does this martingale converge a.s. (with justification)? If yes, what is the limit random variable Y_∞? Give a complete description of the distribution of the RV Y_∞.

Problem 2. Consider a "lazy" random walk on $\mathbb{Z}$ with $S_0 = 0$, evolving as follows: $S_n = \sum_{i=1}^n X_i, n \geq 1$ where $P(X_i = -1) = 1/2$, $P(X_i = 0) = 1/4$ and $P(X_i = 1) = 1/4$. Define the stopping time $\tau = \min(n : S_n = -a \text{ or } b)$, where a, b are positive integers.

(a) Show that τ is finite a.s.

(b) Find $E\tau$.

(c) Find $P(S_\tau = b)$.

Problem 3. Let (A_i) be a sequence of independent events such that $f(n) := \sum_{i=1}^n P(A_i) \to \infty$ as $n \to \infty$. Let $\tau_k = \min(n : \sum_{i=1}^n \mathbb{1}_{A_i} = k)$.

(a) Show that $\tau_k < \infty$ a.s.

(b) Show that $E(f(\tau_k)) = k$ for all $k \geq 1$ (Hint: use OST).

Problem 4.

(a) Give an example of Gaussian random variables X and Y that are uncorrelated (i.e., $\mathrm{Cov}\,(X, Y) = 0$) but not independent.

(b) Let $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ be two independent Gaussian RVs. Show that $X - Y$ and $X + Y$ are independent Gaussian RVs and find their means and variances.

(c) Let (X, Y) be jointly Gaussian RVs with means μ_1, μ_2, variances σ_1^2, σ_2^2, and correlation coefficient ρ. Show that the conditional distribution of X given $Y = y$ is Gaussian, and that $E(X|Y) = \mu_1 + \rho\sigma_1(Y - \mu_2)/\sigma_2$ and the variance $\mathrm{Var}\,(X|Y) = \sigma_1^2(1 - \rho^2)$.

Problem 5. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. Bernoulli RVs with $P(X = 1) = p, P(X = 0) = q$, where $p + q = 1$ and $p \neq q, p > 0, q > 0$. Further, define $S_n = X_1 + X_2 + \cdots + X_n$ and

$$M_n = a_n p^{S_n} q^{n - S_n}, n \geq 1.$$

(a) Find a_n such that $(M_n)_{n\geq 1}$ forms a martingale with respect to the filtration generated by $(X_n)_{n\geq 1}$.

(b) Prove that the sequence (M_n) converges to an a.s. limit and determine the limit.