ENEE620-23. Home assignment 4. Date due November 13, 11:59pm EDT.

Instructor: A. Barg

Please submit your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Papers submitted as multiple pictures of individual pages are difficult for grading and **will not be accepted**.
- Justification of solutions is required.
- Each problem is worth 10 points.

Problem 1. Consider a Galton-Watson process $(X_n)_{n\geq 0}$, where $X_0 = 1$ and the offspring distribution is given by

$$P_Z(m) = \begin{cases} bc^{m-1} & \text{if } m \geq 1 \\ 1 - \frac{b}{1-c} & \text{if } m = 0, \end{cases}$$

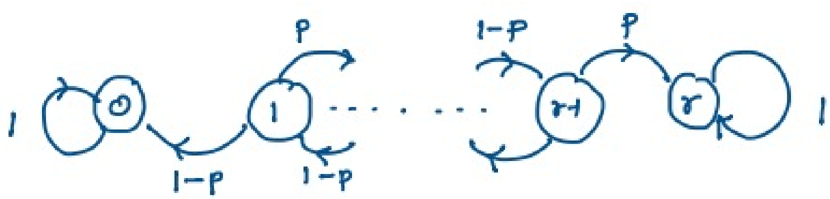
where $b, c > 0$ and $b + c \leq 1$.

(a) Find the generating function $G_Z(z)$ and the expectation of the offspring distribution.

(b) Find the extinction probability of the process.

(c) Show that for any GW process, the expected size of the population in nth generation is $EX_n = (EZ)^n$, and find EX_n for the process considered in this problem.

Problem 2. Consider the Markov chain shown in the figure, where $0 \leq p \leq 1$.



Assume that the chain starts in state $k, 0 < k < r$. We say that the process is absorbed if it enters one of the absorbing states.

(a) Find the absorption probability in state r and the absorption probability in state 0 and argue that the process ends up being absorbed with probability 1.

(b) Find the expected time to absorption for this chain.

Problem 3. (a) Let $X \sim \mathcal{N}(0, 1)$ be a standard Gaussian RV. Is it true that $E(X\mathbb{1}_{\{X>0\}}) = E(X|X > 0)$? Give a yes/no answer, with justification. After that, compute these two quantities.

(b) Let $X \sim \text{Binom}(n, p)$ and $Y \sim \text{Binom}(n, p)$ be independent and assume that $0 < p < 1$. Find $E[X|X + Y]$.

Problem 4. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and put $P(\omega) = 1/6$ for all $\omega \in \Omega$. Let

$$\mathcal{F}_0 = \langle \emptyset, \Omega \rangle, \mathcal{F}_1 = \langle \{4, 5, 6\} \rangle, \mathcal{F} = \langle \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\} \rangle$$

be σ-algebras generated by the collections of sets between the angular brackets $\langle \cdot \rangle$.

(1) Show that the function $X : \Omega \to \mathbb{R}$ defined as $X(i) = \max\{i - 3, 0\}, i \in \Omega$ is a random variable with respect to $\mathcal{F}$ but not with respect to $\mathcal{F}_0, \mathcal{F}_1$.

(2) Find the conditional expectations $E(X|\mathcal{F}_0), E(X|\mathcal{F}_1), E(X|\mathcal{F})$.

Problem 5. Let $(X_n)_n$ be a sequence of independent $\mathcal{N}(0, 1)$ random variables, and let $S_n = X_1 + \cdots + X_n, n \geq 1$. Prove that the sequence

$$Y_n = \frac{1}{\sqrt{n+1}} \exp\left\{\frac{S_n^2}{2(n+1)}\right\}, \quad n \geq 1$$

forms a martingale with respect to the filtration $\mathcal{F} = (\mathcal{F}_n)_{n\geq 1}$, where $\mathcal{F}_n = \sigma(X_1, \ldots, X_n), n \geq 1$.